Example 10-1: Top 100 Films (cont.)
Suppose again that one of the Top 100 films is to be chosen at random, and consider again the original 2×2 table pertaining to Allan and Beth:

<table>
<thead>
<tr>
<th></th>
<th>Beth yes</th>
<th>Beth no</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allan yes</td>
<td>42</td>
<td>6</td>
<td>100</td>
</tr>
<tr>
<td>Allan no</td>
<td>17</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>

a) Given the knowledge that Allan has seen the randomly selected film, what is the conditional probability that Beth has seen it? *Hint:* Restrict your consideration to films that Allan has seen, and ask yourself what fraction of them has Beth seen.

b) How does this conditional probability of Beth having seen the film given that Allan has seen it compare with the (unconditional) probability of Beth having seen the film in the first place? Does the knowledge that Allan has seen the film make it more or less likely (or neither) that Beth has seen it?

c) Suggest how this conditional probability could have been calculated from P(A and B), P(A), and P(B). Which of these three is not needed?

- We denote the *conditional probability* of an event B occurring, given that the event A has occurred, by P(B|A).
- This conditional probability can be calculated as: P(B|A) = P(A and B) / P(A).

d) Use this definition of conditional probability to calculate P(not A | not B), and explain in words what the resulting probability means in this context.

Now consider hypothetical data of the number of these films seen by Chuck and by Donna:

<table>
<thead>
<tr>
<th></th>
<th>Donna yes</th>
<th>Donna no</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chuck yes</td>
<td>15</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Chuck no</td>
<td>45</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>
e) Compare Donna’s (unconditional) probability of having seen the randomly selected film with the conditional probability that Donna has seen it given that Chuck has. Does the knowledge that Chuck has seen the film change the probability that Donna has seen it?

\[ P(D) = \quad P(D|C) = \]

- Two events E and F are said to be **independent** if \( P(E|F) = P(E) \)
  - Otherwise they are said to be **dependent**.
- An equivalent condition for E and F to be independent is that \( P(E \text{ and } F) = P(E) \times P(F) \).

f) Are the events \{Allan has seen the film\} and \{Beth has seen it\} independent? How about \{Chuck has seen the film\} and \{Donna has seen it\}? Explain.

Now suppose you are told that Ellen has seen 80% of the films that Donna has seen.

g) Express this value of .8 as a conditional probability involving the events \( E = \{\text{Ellen has seen it}\} \) and \( D = \{\text{Donna has seen it}\} \).

h) Can you use the information given about Donna and Ellen to determine the proportion of films that have been seen by both Donna and Ellen? If so, please do. [**Hint:** Solve for \( P(D \text{ and } E) \) from the expression for \( P(E|D) \).]

- The **general multiplication rule**, which follows immediately from the definition of conditional probability, asserts that: \( P(A \text{ and } B) = P(A) \times P(B|A) \).
- This can equivalently be written as: \( P(A \text{ and } B) = P(B) \times P(A|B) \).
- When the events are *independent*, this becomes \( P(A \text{ and } B) = P(A) \times P(B) \).

i) In general, is it possible that \( P(E \mid F) \) could have a very different value from \( P(F \mid E) \)?

j) Think more about the previous question by supposing that we choose one American at random. Let \( E = \{\text{person selected is a male}\} \) and \( F = \{\text{person selected is a U.S. Senator}\} \). Make reasonable guesses for \( P(E \mid F) \) and for \( P(F \mid E) \). Are these probabilities similar or very different?
Example 10-2: Graduate School Admissions (cont.)

Suppose that you have applied to two graduate schools and believe that you have a .6 probability of being accepted by school C, a .7 probability of being accepted by school D, and a .5 probability of being accepted by both.

a) Are the events \{acceptance by C\} and \{acceptance by D\} independent? Explain, using two different ways to check for independence.

a2) Are the events \{acceptance by C\} and \{acceptance by D\} mutually exclusive (disjoint)? Explain.

b) Determine the conditional probability of acceptance by D given acceptance by C. How does it compare to the (unconditional) probability of acceptance by D? In other words, learning that you have been accepted by school C makes it _____ likely that you have been accepted by school D.

Now suppose that you change your mind and apply to two different schools E and F. You regard the events \{acceptance by E\} and \{acceptance by F\} to be independent, with probability of acceptance by E equal to .8 and probability of acceptance by F equal to .3.

c) Determine the probability of acceptance by both schools (E and F). Then determine the probability of acceptance by at least one school (E or F).

Accepted by both:

Accepted by at least one:

- **Multiplication rule** for a series of independent events $E_1$, $E_2$, ..., $E_k$:
\[
P(E_1 \text{ and } E_2 \text{ and } \ldots \text{ and } E_k) = P(E_1) \times P(E_2) \times \ldots \times P(E_k).
\]

Suppose that you also apply to graduate schools G and H, that you consider all acceptances to be independent of each other, and that you believe the probabilities of acceptance to be .9 and .5, respectively.

d) Determine the probability of acceptance by all four schools (E and F and G and H).
e) Determine the probability of acceptance by at least one of the four schools (E or F or G or H). 

*Hint:* First find the probability of the complement of this event.

f) Suppose that you have a .7 probability of being accepted by a school and that given your acceptance, the conditional probability of receiving financial aid is .9. Determine the probability that you are both accepted and receive financial aid from this school. (First ask if these events are independent.)

**Example 10-3: Daily Lottery**
Suppose that you play a daily lottery game in which you bet on one 3-digit number, where each digit is equally likely to be any number in 0 – 9. Let \( W_i \) denote the event that you win on day \( i \), and let \( L_i \) denote the event that you lose on day \( i \).

Are the \( W_i \)'s independent? Are they mutually exclusive?

a) Determine the probability that you win on the very first day that you play.

b) Determine the probability that you win at least once if you play every day for a 7-day week. Also express this probability in terms of the events \( W_i \) and \( L_i \), and be clear about what probability rule(s) you use to calculate this. Record your answer to five decimal places.

c) Repeat b) if you play every day for a 365-day year.

d) Suppose that a friend tells you that you are guaranteed to win at least once if you play for 1000 days. How would you respond?

e) Calculate the probability that you would win at least once if you play for 1000 days.
Example 10-4: Unfinished Game (cont.)
Reconsider the unfinished game: Heather and Tom play a game that involves a series of coin flips. They agree that if 5 heads occur before 5 tails do, then Heather wins the game. But if 5 tails occur before 5 heads do, then Tom wins the game. They each agree to pay $5 to play the game, so the winner will make a profit of $5. The first five coin tosses result in: H₁, T₂, T₃, H₄, H₅. Unfortunately, the game is interrupted at that point and can never be finished.

a) Now suppose that the game is to be finished, beginning with the 6th flip. Reproduce the sample space of all possible outcomes.

b) Use independence and the multiplication rule to determine the probability for each of these outcomes.

c) Determine the probability that Heather would win the game if it were continued. Also determine the probability that Tom would win the game if it were continued.

d) Based on these probabilities, how should the $10 that the players contributed be distributed?