One of the most important and useful concepts in probability is independence. The basic idea is that events are independent if learning about the occurrence (or non-occurrence) of one event does not change the probability of the other event. The following three conditions are all equivalent definitions:

- Two events $E$ and $F$ are independent if:
  - $\Pr(E|F) = \Pr(E)$
  - $\Pr(F|E) = \Pr(F)$
  - $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$
  - This last equality is called the multiplication rule for independent events.
- Two events that are not independent are said to be dependent.

Sometimes we check whether given probabilities imply independence between events, and sometimes we assume that events are independent so we can calculate the probability of their intersection by multiplying their individual probabilities.

**Example 6-1: Top 100 films (cont.)**
Recall again the following table:

<table>
<thead>
<tr>
<th></th>
<th>Beth yes</th>
<th>Beth no</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allan yes</td>
<td>42</td>
<td>6</td>
<td>48</td>
</tr>
<tr>
<td>Allan no</td>
<td>17</td>
<td>35</td>
<td>52</td>
</tr>
<tr>
<td>Total</td>
<td>59</td>
<td>41</td>
<td>100</td>
</tr>
</tbody>
</table>

Suppose again that a film is chosen at random. Are the events $\{\text{Allan has seen the film}\}$ and $\{\text{Beth has seen the film}\}$ independent? Justify your answer in three different ways.

**Example 6-2: Rolling Dice (cont.)**
a) Roll two fair, six-sided dice. Which pairs of the following events are independent and which are not?

- $A = \{\text{sum is 7}\}$
- $B = \{\text{sum is 11}\}$
- $C = \{1^{\text{st}} \text{ number is 6}\}$
- $D = \{\text{second number is 3}\}$
b) Notice that events A and B are mutually exclusive. What can you conclude about whether events that are mutually exclusive are also independent? Explain.

- General multiplication rule for independent events: If $E_1, E_2, \ldots, E_k$ are independent events, then $\Pr(E_1 \cap E_2 \cap \ldots \cap E_k) = \Pr(E_1) \times \Pr(E_2) \times \Pr(E_3) \times \ldots \times \Pr(E_k)$.

**Example 6-3: Graduate School Applications (cont.)**
Suppose that you have applied to two graduate schools E and F, and you assess your probability of being accepted by E as .6 and by F as .7. You also consider these to be independent events.

a) Determine the probability that you are accepted by both schools.

b) Determine the probability that you are accepted by at least one school.

Now suppose that you have also applied to graduate schools G and H, with acceptance probabilities of .4 and .9, respectively. You consider all events to be independent of each other.

c) Determine the probability that you are accepted by all four schools.

d) Determine the probability that you are accepted by at least one of the four schools. [Hint: First find the probability of the complement.]
Example 6-4: Daily Lottery
Suppose that every day a person plays a lottery game that has a 1/1000 probability of winning. Determine the probability that the person wins at least once if he plays every day for:

a) a 7-day week (also explain why the answer is not exactly .007)

b) a 31-day month

c) a 365-day year

d) Determine and graph the probability of winning at least once, as a function of the number of days \( n \). Describe the behavior of this function.

e) For how many days must the person play to have probability > .9 of winning at least once? [Determine the answer both analytically and by examining the graph.]

Example 6-5: Unfinished Game (cont.)
Reconsider Example 1-5. Recall that Heather and Tom play a game that involves a series of coin tosses. They agree that if 5 heads occur before 5 tails do, then Heather wins the game. But if 5 tails occur before 5 heads do, then Tom wins the game. They each to pay $5 to play the game, so the winner will take all $10 for a profit of $5. The first five tosses produce: H1, T2, T3, H4, H5. Unfortunately, the game is interrupted at that point and can never be finished.
a) List the outcomes in the event that Heather would go on to win the game if it were continued from that point. (Use similar notation as above, so one such outcome is H6H7.)

b) Determine the probability of each of the outcomes in a).

c) Determine the probability that Heather would win if the game were to be continued from the point of interruption.

d) If the $10 were distributed according to the probabilities of winning from the point of interruption, how much would Heather get?

e) Determine the probability that the game requires four more flips.

Example 6-6: Foul Shooting Contest
Suppose that Janelle has a .4 probability of making a foul shot, and Kaesha has a .8 probability. Because Kaesha is the better shooter, she lets Janelle shoot first, and then they alternate until one of them makes a shot. Whoever makes a shot first wins the contest.

a) Before doing any calculations, make a guess for the probability that Janelle wins this contest.

b) Describe what assuming independence means in this context.
c) Determine the probability that Janelle wins this contest on her first shot.

d) Determine the probability that Janelle wins this contest on her second shot. [Hint: Think about what has to happen on earlier shots for both players.]

e) Determine the probability that Janelle wins this contest on her third shot.

f) Use an infinite sum to determine the probability that Janelle wins this contest.

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**Example 6-7: Unusual Dice**

Consider four six-sided dice with unusual numbers on their sides, as follows:

- Die A: 4, 4, 4, 4, 0, 0
- Die B: 3, 3, 3, 3, 3, 3
- Die C: 6, 6, 2, 2, 2, 2
- Die D: 5, 5, 5, 1, 1, 1

Assume you and I choose two dice to roll independently, and the one producing the larger number wins. Being considerate, I’ll ask you to choose your die first.

a) For whichever pair of dice that you and I choose, determine the probability that the number you roll will be larger than the number that I roll.

b) Suggest a name for this example: Non- ___________ Dice.