Example 4-1: Top 100 Films (cont.)
Suppose again that one of the Top 100 films is to be chosen at random, and consider again the original 2×2 table pertaining to Allan and Beth:

<table>
<thead>
<tr>
<th></th>
<th>Beth yes</th>
<th>Beth no</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allan yes</td>
<td>42</td>
<td>6</td>
<td>100</td>
</tr>
<tr>
<td>Allan no</td>
<td>17</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>

a) Given the knowledge that Allan has seen the randomly selected film, what is the conditional probability that Beth has seen it? [Hint: Restrict your consideration to films that Allan has seen, and ask yourself what fraction of them has Beth seen.]

b) How does this conditional probability of Beth having seen the film given that Allan has seen it compare with the (unconditional) probability of Beth having seen the film in the first place? Does the knowledge that Allan has seen the film make it more or less likely (or neither) that Beth has seen it?

c) Suggest how this conditional probability could have been calculated from \( P(A \text{ and } B) \), \( P(A) \), and \( P(B) \). Which of these three is not needed?

- We denote the conditional probability of an event \( E \) occurring, given that the event \( F \) has occurred, by \( P(E|F) \).
- This conditional probability can be calculated as: \( P(E|F) = P(E \cap F) / P(F) \).

d) Use this definition of conditional probability to calculate \( P(A^c | B^c) \), and explain in words what the resulting probability means in this context.

Now consider hypothetical data of the number of these films seen by Chuck and by Donna:

<table>
<thead>
<tr>
<th></th>
<th>Donna yes</th>
<th>Donna no</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chuck yes</td>
<td>15</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Chuck no</td>
<td>45</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>

e) Compare Donna’s (unconditional) probability of having seen the randomly selected film with the conditional probability that Donna has seen it given that Chuck has. Does the knowledge that Chuck has seen the film change the probability that Donna has seen it?

\[
P(D) = \quad P(D|C) =
\]
Two events E and F are said to be **independent** if P(E | F) = P(E)

- Otherwise they are said to be **dependent**.

An equivalent condition for E and F to be independent is that P(E and F) = P(E) × P(F).

f) Are the events \{Allan has seen the film\} and \{Beth has seen it\} independent? How about \{Chuck has seen the film\} and \{Donna has seen it\}? Explain.

Now suppose you are told that Ellen has seen 80% of the films that Donna has seen.

g) Express this value of .8 as a conditional probability involving the events E = \{Ellen has seen it\} and D = \{Donna has seen it\}.

h) Use the information given about Donna and Ellen to determine the proportion of films that have been seen by *both* Donna and Ellen? [Hint: Solve for P(D ∩ E) from the expression for P(E | D).]

- The **general multiplication rule**, which follows immediately from the definition of conditional probability, asserts that: P(A ∩ B) = P(A) × P(B | A).
  - This can equivalently be written as: P(A ∩ B) = P(B) × P(A | B).
  - When the events are independent, this becomes P(A ∩ B) = P(A) × P(B).

i) In general, is it possible that P(E | F) could have a very different value from P(F | E)?

j) Think more about the previous question by supposing that we choose one American at random. Let E = \{person selected is a male\} and F = \{person selected is a U.S. Senator\}. Make reasonable guesses for P(E | F) and for P(F | E). Are these probabilities similar or very different?

**Example 4-2: Graduate School Admissions (cont.)**

Suppose again that you have applied to two graduate schools and believe that you have a .6 probability of being accepted by school C, a .7 probability of being accepted by school D, and a .5 probability of being accepted by both.

a) Are the events \{acceptance by C\} and \{acceptance by D\} independent? Explain, using two different ways to check for independence.

b) Are the events \{acceptance by C\} and \{acceptance by D\} mutually exclusive (disjoint)? Explain.
c) Determine the conditional probability of acceptance by D given acceptance by C. How does it compare to the (unconditional) probability of acceptance by D?

Now suppose that you change your mind and apply to two different schools E and F. You regard the events \{acceptance by E\} and \{acceptance by F\} to be **independent** (although this is probably not realistic), with probability of acceptance by E equal to .8 and probability of acceptance by F equal to .3.

d) Determine the probability of acceptance by both schools (E and F). Then determine the probability of acceptance by at least one school (E or F).

\[
\text{Accepted by both:} \\
\text{Accepted by at least one:}
\]

- **Multiplication rule** for a series of independent events $E_1, E_2, \ldots, E_k$:
  \[
P(E_1 \text{ and } E_2 \text{ and } \ldots \text{ and } E_k) = P(E_1) \times P(E_2) \times \ldots \times P(E_k).
\]

Suppose that you also apply to graduate schools G and H, that you consider all acceptances to be independent of each other, and that you believe the probabilities of acceptance to be .9 and .5, respectively.

e) Determine the probability of acceptance by all four schools (E and F and G and H).

f) Determine the probability of acceptance by at least one of the four schools (E or F or G or H).  
   [**Hint:** First find the probability of the complement of this event.]

**Example 4-3: Daily Lottery**
Suppose that you play a daily lottery game in which you bet on one 3-digit number, where each digit is equally likely to be any number in $0 – 9$. Let $W_i$ denote the event that you win on day $i$, and let $L_i$ denote the event that you lose on day $i$.

a) Are the $W_i$'s independent? Are they mutually exclusive? Explain.
b) Determine the probability that you win on the very first day that you play.

c) Determine the probability that you win at least once if you play every day for a 7-day week. Also express this probability in terms of the events $W_i$ and $L_i$, and be clear about what probability rule(s) you use to calculate this. [Hint: Use the complement rule.] Also explain why this probability does not equal .007.

d) Determine the probability that you win at least once if you play every day for a 30-day month. Then repeat this calculation if you play every day for a 365-day year.

e) Suppose that a friend tells you that you are guaranteed to win at least once if you play for 1000 days. How would you respond?

f) Express the probability of winning at least once as a function of the number of days that you play (call this $n$). Use Excel to graph this probability as a function of $n$. Comment on the shape of this function.

g) Determine how many days you would have to play in order to have a greater than .9 probability of winning at least once. [Hint: Use logs, and verify your answer from the graph.]

Example 4-4: Unfinished Game (cont.)
Recall that Heather and Tom play a game that involves a series of coin tosses. They agree that if 5 heads occur before 5 tails do, then Heather wins the game. But if 5 tails occur before 5 heads do, then Tom wins the game. They each to pay $5 to play the game, so the winner will take all $10 for a profit of $5. The first five tosses produce: $H_1, T_2, T_3, H_4, H_5$. Unfortunately, the game is interrupted at that point and can never be finished.
a) List the outcomes in the event that Heather would go on to win the game if it were continued from that point. (Use similar notation as above, so one such outcome is H₆H₇.)

b) Determine the probability of each of the outcomes in a).

c) Determine the probability that Heather would win if the game were to be continued from the point of interruption.

d) If the $10 were distributed according to the probabilities of winning from the point of interruption, how much would Heather get?

e) Determine the probability that the game requires four more flips.

**Example 4-5: Unusual Dice**

Consider four six-sided dice with unusual numbers on their sides, as follows:

- Die A: 4, 4, 4, 4, 0, 0
- Die B: 3, 3, 3, 3, 3, 3
- Die C: 6, 6, 2, 2, 2, 2
- Die D: 5, 5, 5, 1, 1, 1

Assume you and I choose two dice to roll independently, and the one producing the larger number wins. Being considerate, I’ll ask you to choose your die first.

a) For whichever pair of dice that you and I choose, determine the probability that the number you roll will be larger than the number that I roll.

b) Suggest a name for this example: Non-__________ Dice.