STAT 325 – Handout 13
Transition and Steady-State Probabilities (4.3, 4.6)

Example 13-1: Lunch (cont.)
Recall the example that every day I have either a taco or a slice of pizza for lunch. If I get a slice of pizza on one day, then I switch to a taco the next day with probability .8 and I have pizza again with probability .2. If I get a taco on one day, then I have a taco again the next day with probability .4 and I switch to pizza with probability .6. Thus, the one-step transition probability matrix is: $P = \begin{pmatrix} .2 & .8 \\ .6 & .4 \end{pmatrix}$, where the first row/column represent a slice of pizza and the second row/column represent a taco.

a) Suppose that I have a slice of pizza on Monday. What are the probabilities for my lunch entrée on Tuesday?

b) Still suppose that I have a slice of pizza on Monday. What is the probability that I have pizza for lunch on *Wednesday*? [*Hint*: Apply the law of total probability, conditioning on what I have for lunch on Tuesday.]

c) Now determine the probability that I have a taco on Wednesday, still supposing that I have a slice of pizza on Monday.

d) Now suppose that I have a taco on Monday. Determine the probabilities of my two lunch possibilities for Wednesday.

e) Organize the four probabilities that you have calculated in b-d) into a matrix.

- The **2-step transition probability matrix** contains $\Pr[X(n+2) = j \mid X(n) = i]$.
- The ***k*-step transition probability matrix** contains $\Pr[X(n+k) = j \mid X(n) = i]$.

f) How does your matrix in e) compare to the original (1-step) matrix P?

- Key result: The *k*-step transition probability matrix turns out to equal the ______________ of the one-step transition probability matrix.

g) Use R to calculate P^2 and verify this result.

```
P = rbind(c(.2,.8), c(.6,.4))
P2 = P %*% P
P2
```

h) Use R to calculate P^3 and P^4. Also interpret what the probabilities in these matrices mean.

```
P3 = P2 %*% P
P4 = P2 %*% P2
P3; P4
```

i) Use R to calculate P^8, P^{16}, P^{32}, and P^{64}. What do you notice?

j) Is what you found in i) consistent with our earlier simulation results? Explain.

k) Now suppose that I flip a coin to determine what to have for lunch on Monday, and then I follow the probabilities of this Markov chain. What then are the probabilities of my lunch options for Tuesday?

- The probabilities of starting the process in the various states can be organized into a row vector called the **initial probability vector**, denoted by π_0.
- The probabilities of being in the various states after k steps can then be organized into a row vector denoted by π_k and calculated as $\pi_k = \pi_0 \times P^k$.

l) Suppose again that I flip a coin to determine what to have for lunch on the first day. What are the probabilities of my lunch options 64 days later? How do these compare to what you found above, when I started with a taco or pizza for sure on the first day?

```
pi0 = rbind(c(.5,.5))
pi64 = pi0 %*% p64
```

Example 13-2: Ping-pong tournament (cont.)

Recall the ping-pong example: Whoever loses the current game steps aside for the next game, allowing the winner of the current game and the player currently sitting out to play in the next game. Suppose that player A has probability .7 of beating B and probability .8 of beating C in a game, and B has probability .6 of beating C in a game with results of all games being independent of each other.

a) Report the 1-step transition probability matrix.

b) If the 1st game is between players A and B, determine the probability that the 2nd game will also be between players A and B.

c) If the 1st game is between players A and B, determine the probability that the 3rd game will also be between players A and B. First answer this without using matrix algebra, and then confirm your answer with matrix algebra.

d) If the 1st game is between players A and B, determine the probability that the 4rd game will also be between players A and B.

e) If the 1st game is between players A and B, determine the probability that the 5th game will also be between players A and B.

f) If the 1st game is between players A and B, determine the probability that the 65th game will also be between players A and B.

g) How different is the probability that the 65th game will be between players A and B, depending on who plays the 1st game? Explain.

Example 13-3: Umbrellas (cont.)
Reconsider the umbrella example: I own 4 umbrellas, and I start by keeping 2 at home and 2 at my office. I only take an umbrella with me when it's actually raining. There's a 20% chance of rain when I leave home every morning, independently from day to day, and a 30% chance of rain when I leave work each evening, independently from day to day and also independent of the weather at home that morning.

a) Report the 1-step transition probability matrix P.

b) Determine the expected number of umbrellas that I have at home after 1 day.

c) After 2 days, what is the probability distribution of the number of umbrellas that I have at home? Indicate how to calculate this from the matrix P.

d) Determine the expected number of umbrellas that I have at home after 2 days.

e) After 64 days, what is the probability distribution of the number of umbrellas that I have at home?

f) Determine the expected number of umbrellas that I have at home after 64 days.

g) How do the expected values in b) and d) and f) compare? Explain why this makes sense.

Example 13-4: Lunch (cont.)
Recall the example that every day I have either a taco or a slice of pizza for lunch. The one-step transition probability matrix is: $P = \begin{pmatrix} .2 & .8 \\ .6 & .4 \end{pmatrix}$.

a) Report the following matrices: P^8, P^{16}, and P^{32}.

b) What do these matrices reveal?

- A Markov chain is said to be **regular** if the *k*-step transition matrix has all non-zero entries for some value of *k*.
- With a regular Markov chain, as $k \rightarrow \infty$, the rows of P^k converge to identical rows with non-zero entries. This row vector is denoted by $\boldsymbol{\pi}$ and called the **steady-state probability vector**.
    - The steady-state probability vector indicates the long-run probability distribution of the various states as the process continues for a very long time.
    - The steady-state probability vector is the long-run expected fraction of visits that the Markov chain makes to the various states.
    - The steady-state probability vector can be approximated by simulation or by calculating P^k for a large value of *k*.

c) Based on the matrices above, indicate the steady-state probability vector for the pizza/taco lunch Markov chain.

- The steady-state probability vector π can be determined (exactly) by solving the following system of equations: $\boldsymbol{\pi}P = \boldsymbol{\pi}$, $\pi_1 + \pi_2 + \ldots + \pi_s = 1$

d) Solve this system of equations for the pizza/taco lunch Markov chain.

e) Interpret what this steady-state probability vector means. Is this vector consistent with our simulation results from the other day?

Example 13-5: Ping-pong tournament (cont.)
a) Set up and solve the appropriate system of equations to determine the steady-state probability vector for the ping-pong tournament.

b) Interpret what these steady-state probabilities mean.

c) Conduct a simulation with 100,000 games. Are the results within the margin-of-error of the steady-state probabilities?

Example 13-6: Conversation

Suppose that three friends Daphne, Emma, Francine engage in a conversation in which only one speaks at a time. When one of them stops speaking, a different person begins to speak, according to the following Markov chain transition probabilities:

	To Daphne	To Emma	To Francine
From Daphne	0	.8	.2
From Emma	.6	0	.4
From Francine	.3	.7	0

a) Based on these transition probabilities, who do you think will speak the most in the long run? Who do you think will speak the least? Explain.

b) Calculate P^{32}, and see if that approximates the steady-state probability vector.

c) Set up and solve the appropriate system of equations to determine the (exact) steady-state probability vector.

d) Interpret what the steady-state probability vector reveals.