STAT 325 – Handout 14
Absorbing States (4.7)

We have been studying Markov chains, and we have found that *regular* Markov chains approach steady-state probabilities as the process continues. Today we investigate the behavior of Markov chains that are not regular, meaning that when they reach a particular state, they never leave that state.

Example 14-1: Roulette (cont.)
Suppose you have \$3 but you desperately need to have \$5. You decide to play roulette to try to increase your \$3 into a \$5 fortune. You bet \$1 on a color and continue to do this until you have either \$5 or \$0.

a) Determine the 1-step transition probability matrix for the amount of money that you have after any spin of the wheel. (Recall that your probability of winning \$1 with this bet is 18/38, and your probability of losing \$1 with this bet is 20/38.)

- A state is called **absorbing** if $\Pr[X(n) = j | X(n-1) = j] = 1$.
    - Otherwise, a state is called *non-absorbing*.

b) Which states in this process are absorbing? Which are non-absorbing?

- One question of interest is how many steps it takes before the process enters an absorbing state, which is called *time to absorption*.

Denote the number of roulette spins that you take in this game (before reaching either \$0 or \$5) by the random variable T.

c) What is the smallest possible value of T? Determine its probability, without using the probability matrix.

d) Calculate P^2. Does the answer to c) appear in this matrix? Explain why this makes sense.

e) Now determine the probability for the next smallest possible value of T, using probability rules and not matrix algebra. Then calculate P^3 and identify where the answer appears in that matrix.

f) Calculate P^4. What does this matrix tell you about the probability distribution of T? Explain.

g) Repeat e) for P^8 and P^{32}.

- Of particular interest is the *mean time to absorption* from the various initial states.
    - We can approximate these mean times by simulation.
    - We can also (in principle) calculate $E(T) = \sum_{t=1}^{\infty} t \Pr(T = t)$, where $\Pr(T = t)$ is calculated by $\Pr(T \le t) - \Pr(T \le t - 1)$, where these in turn are determined from the matrices P^t and P^{t-1}.
    - We can calculate these mean times using a matrix formula derived in your text (Theorem 4.7.2, page 180).
        - Number the nonabsorbing states as 1, 2, …, r.
        - Denote the mean times to absorption of those states by $\mu_1, \mu_2, \ldots, \mu_r$.
        - Let Q represent the 1-step transition probability matrix for the nonabsorbing states.
        - The mean times to absorption satisfy the system of equations: $(I - Q)\boldsymbol{\mu} = \mathbf{1}$
            - Where I is the $r \times r$ identity matrix, $\boldsymbol{\mu}$ is the $r \times 1$ vector of mean times, and $\mathbf{1}$ is the $r \times 1$ vector of ones.
            - This equation can be solved for $\boldsymbol{\mu}$ using the `solve` command in R.

h) How many nonabsorbing states are there in this roulette example?

i) Determine the matrix Q for this roulette example.

j) Determine the matrix (I-Q) and set up the system of equations to be solved.

k) Use R to solve for the vector of mean times to absorption:

```
# first enter matrix Q
I = diag(4)
ones = rep(1, times = 4)
meantimes = solve(I-Q, ones)
meantimes
```

l) Interpret these mean times to absorption in terms of expected number of roulette spins.

m) Which state has the largest mean time to absorption? Which has the shortest? Do these make sense?

- Another question of interest is the probability of ending up in a particular absorbing state as opposed to another absorbing state.
  - We can approximate this probability by simulation.
  - We can approximate this probability by calculating the *k*-step transition probability matrix P^k for a large number *k*.
  - We can calculate this probability exactly with a matrix formula derived in your text (Theorem 4.7.3, page 184).
    - Let F be a matrix with a row for each nonabsorbing state and a column for each absorbing state, with elements of F giving probability of eventually reaching (absorbing) state *j* if you start in (nonabsorbing) state *i*.
    - Re-arrange the 1-step transition probability matrix P in terms of submatrices I, Q, and R as shown on page 184.
    - The matrix F is the solution to (I-Q)F = R.

n) Approximate these probabilities for our roulette example by calculating P^{64}.

o) Re-arrange P into the desired form, and identify the matrices I, Q, and R.

p) Use R to solve for the matrix F:

```
F = solve(I-Q, R); F
```

q) Interpret what each element of the matrix F represents.

r) With your starting amount of \$3, do you have a better than 50/50 chance of reaching your goal of \$5 before you go broke at \$0?