STAT 325 – Handout 17
Exponential Distribution (6.1)

- A random variable X is said to have an **exponential** distribution with parameter λ (where λ > 0) if the pdf of X is: $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f(x) = 0$ for $x \le 0$.
  - Notation: X ~ Expo(λ)
  - Minitab calculations: `Graph> Probability Distribution Plot> Exponential`
    - Scale parameter = 1/λ; Threshold parameter = 0
  - R: pmf: `dexp(x, λ)`; cdf: `pexp(x, λ)`, inverse cdf: `qexp(prob, λ)`, simulation: `rexp(numreps, λ)`

Exmaple 17-1: Some Derivations

a) Determine the cdf of an exponential distribution with parameter λ.

b) Determine the expected value of an exponential random variable. [*Hint*: Use integration by parts.]

c) Determine the variance and standard deviation of an exponential random variable. [*Hint*: Use the following calculus result, which follows from repeated application of integration by parts: $\int_0^\infty x^n \lambda e^{-\lambda x} dx = \frac{n!}{\lambda^n}$.]

Example 17-2: Fast-food service time

Suppose that the service time (the time between placing an order and receiving food) for a randomly selected customer at a particular fast-food restaurant follows an exponential distribution with mean 1.25 minutes. Let the random variable T represent this service time.

a) Determine and sketch the pdf of T.

b) Determine the probability that this service time is less than 1 minute. Show how to use the pdf or cdf to calculate this.

c) Determine the probability that this service time is more than 2 minutes.

d) Determine the value (call it *m*) such that the probability is .5 that the service time will be less than or equal to *m*.

e) Determine the value such that only 5% of all service times exceed that value.

f) Use Minitab to verify the calculations above.

g) Use R to simulate 100,000 repetitions of an exponential random variable with parameter λ = 0.8. Then use the simulation results to investigate the reasonableness of your answers above.

```
times = rexp(100000, 0.8)
hist(times)
probb = (times < 1)
table(probb)
```

h) Given that you have already been waiting 2 minutes to be served, determine the probability that you must wait for an additional 2 minutes or more. What do you notice about your answer?

i) Indicate how to use your R simulation to approximate the probability in h). Then do that and report the approximate probability. Is it similar to your answer to h)?

More generally, let x and t be any positive real numbers, and let the exponential parameter be λ.

j) Determine $\Pr(X > x + t \mid X > x)$. What do you notice about the result?

- Let $X \sim \text{Expo}(\lambda)$. Then X satisfies the *memoryless* property.
  - For positive real numbers x and t, $\Pr(X > x + t \mid X > x) = \Pr(X > t)$.
  - In other words, the process begins anew at every moment.

Suppose that three customers place their orders at the same time. Let X_i represent the service time for customer i, and assume that the X_i's are independent with $X_i \sim \text{Expo}(0.8)$.

k) Determine the probability that at least one of these customers is served in less than 1 minute. [*Hint*: First find the probability of the complement. Think about how to express the complement in terms of X_1, X_2, and X_3.]

l) Use an R simulation to approximate the probability in k).

Now let Y represent the time until one of these customers is served, so $Y = \min\{X_1, X_2, X_3\}$.

m) For any positive value y, determine $\Pr(Y > y)$. [*Hint*: Think about what must be true about X_1, X_2, and X_3 in order for Y to be greater than y.]

n) Use your previous answer to determine the cdf of Y.

o) Use your previous answer to determine the pdf of Y. Does the form of this pdf look familiar? Explain.

- Let $X_1, X_2, \ldots X_n$ be independent random variables with $X_i \sim \text{Expo}(\lambda)$, and let the random variable $Y = \min\{X_i\}$. Then $Y \sim \text{Expo}(n\lambda)$.