STAT 325 – Handout 2
Probability Axioms, Basic Rules (1.2)

Recall that:

- The **sample space** (S) of a random process is a set consisting of all possible outcomes.
- An **event** is a subset of the sample space.
- Two events are **mutually exclusive** if they have no outcomes in common (i.e., if their intersection is the empty set).

Now consider that:

- A **probability** is a *function* that takes sets as inputs and gives real numbers as outputs.

The assignment of probabilities to events must satisfy three *axioms*:

1. For any event A, $0 \leq \Pr(A) \leq 1$.
2. $\Pr(S) = 1$, where S denotes the sample space.
3. If $E_1, E_2, \ldots, E_n, \ldots$ are mutually exclusive events, then $\Pr(E_1 \cup E_2 \cup \ldots \cup E_n \cup \ldots) = \Pr(E_1) + \Pr(E_2) + \ldots + \Pr(E_n) + \ldots$.

Example 2-1. Top 100 Films

In 1998 the American Film Institute created a list of the top 100 American films ever made (`www.afi.com/tvevents/100years/movies.aspx`). Suppose that one of these 100 movies is selected at random.

Notation: Let A denote the subset of these 100 films that Allan has seen, so the event A = {Allan has seen the film}. Similarly define events B and F for Beth and Frank, respectively. Note that:

- A^c denotes the complement of A and is interpreted as "not A"
- $A \cup B$ denotes the union of A and B and is interpreted as "A or B," which is equivalent to "A or B or both"
- $A \cap B$ (also written as simply AB) denotes the intersection of A and B and is interpreted as "A and B."

Notice that *events* are *sets*. (In particular, they are subsets of the sample space S.) Thus, it is legitimate to perform set operations such as complement, intersection, and union on them. On the other hand, *probabilities* are *numbers*. More specifically, they are numbers between 0 and 1 (including those extremes). Thus, it is legitimate to add, multiply, and divide probabilities but not to take complements, intersections, or unions of them.

The "at random" selection implies that each of the 100 films is equally likely to be chosen (i.e., each has probability 1/100). Thus, the probabilities of these various events can be calculated by counting how many of the 100 films comprise the event of interest. For example, the following 2×2 table classifies each movie according to whether it was seen by Allan and whether it was seen by Beth:

	Beth yes	Beth no	Total
Allan yes	42	6	
Allan no	17	35	
Total			100

a) Translate the following events into set notation using the symbols A and B, complement, union, intersection. Also give the probability of the event as determined from the table:

Event in words	Event in set notation	Probability
Allan and Beth have both seen the film		
Allan has seen the film and Beth has not		
Beth has seen the film and Allan has not		
Neither Allan no Beth has seen the film		

b) Fill in the *marginal* totals of the table (the row and column totals). From these totals determine the probability that Allan has seen a randomly selected film and also the probability that Beth has seen the film. (Remember that the film is chosen at random, so all 100 are equally likely.) Record these, along with the appropriate symbols, below.

Pr(Allan has seen it) = P( ) =

Pr(Beth has seen it) = P( ) =

c) Determine the probability that Allan has *not* seen the film. Do the same for Beth. Record these, along with the appropriate symbols, below.

d) If you had not been given the table, but instead had merely been told that Pr(A)=.48 and Pr(B)=.58, would you have been able to calculate $Pr(A^c)$ and $Pr(B^c)$? Explain how.

- **Complement rule**: For any event A, $Pr(A^c) = 1 - Pr(A)$

e) Add the counts in the appropriate cells of the table to calculate the probability that either Allan or Beth (or both) have seen the movie. Also indicate the symbols used to represent this event.

f) If you had not been given the table but instead had merely been told that Pr(A) = .48 and Pr(B) = .59, would you have been able to calculate $Pr(A \cup B)$? Explain.

g) One might naively think that $Pr(A \cup B) = Pr(A) + Pr(B)$. Calculate this sum, and indicate whether it is larger or smaller than $Pr(A \cup B)$ and by how much. Explain why this makes sense, and indicate how to adjust the right side of this expression to make the equality valid. Also use a *Venn diagram* to represent this.

- **Addition rule**: $Pr(A \cup B) = Pr(A) + Pr(B) – Pr(A \cap B)$.

h) Use this addition rule as a second way to calculate the probability that Allan or Beth has seen the movie, verifying your answer to (e).

i) As a third way to calculate this probability, first identify (in words and in symbols) the *complement* of the event {Allan or Beth has seen the movie}. Then find the probability of this complement from the table. Then use the complement rule to determine $P(A \cup B)$. Are your answers to (e) and (h) confirmed?

j) What has to be true about A and B in order for $P(A \cup B) = P(A) + P(B)$?

Now consider Frank, who has seen 61 of the top 100 films. The following pair of 2×2 tables (which can be considered a 2×2×2 table) reveal the counts. The 2×2 table on the left pertains to films Allan has seen, and the 2×2 table on the right pertains to films Allan has not seen:

Allan yes	Frank yes	Frank no	Total	Allan no	Frank yes	Frank no	Total
Beth yes	40	2	42	Beth yes	13	4	17
Beth no	3	3	6	Beth no	5	30	35
Total	43	5	48	Total	18	34	52

k) What is the probability that all three of them have seen the film? Also indicate the symbols used to denote this event.

l) Express in symbols and in words the event for which the probability can be read directly from the table to be 13/100.

We will again find three ways to calculate the probability that at least one of these three people has seen the film.

m) Express this event (that at least one of these three people has seen the film) in symbols.

n) Determine $P(A \cup B \cup F)$ directly from the table by adding the counts of the outcomes that comprise this event.

o) Express the complement of this event (that at least one of these three people has seen the film) in words and in symbols. Use the table to determine the probability of this complement. Then use the complement rule to find the probability that at least one of the three has seen the film. Does the answer agree with (n)?

p) Suppose that instead of being given the tables, you had only been told that Allan has seen 48 of the films, Beth 59, and Frank 61. Would that information alone enable you to determine the number of films that at least one of these three has seen? Explain.

q) To see how to calculate $P(A \cup B \cup F)$ from other probabilities, construct a *Venn diagram* with overlapping circles representing the films that Allan, Beth, and Frank have seen.

Addition rule for three events:

- $P(A \cup B \cup C) = P(A) + P(B) + P(C) – P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Example 2-2. Graduate School Admissions
Suppose that you have applied to two graduate schools and believe that you have a .6 probability of being accepted by school E, a .7 probability of being accepted by school F, and a .5 probability of being accepted by both.

a) Organize this information into a probability table. Also fill in the rest of the table.

b) Represent these probabilities in a Venn diagram.

c) Determine the probability of being accepted by at least one of these two schools. Use the addition rule, and verify your answer with the table and Venn diagram.

d) Determine the probability of being rejected by both schools. Also use appropriate notation to describe this event.

e) Determine the probability of accepted by one school but not both schools. Also use appropriate notation to describe this event.

f) Consider the event $E^c \cup F^c$. Describe this event in words, and calculate its probability using the complement rule.

DeMorgan's laws:

- $(E \cap F)^c = E^c \cup F^c$
- $(E \cup F)^c = E^c \cap F^c$
- $(E_1 \cap E_2 \cap \ldots \cap E_k)^c = E_1^c \cup E_2^c \cup \ldots \cup E_k^c$
- $(E_1 \cup E_2 \cup \ldots \cup E_k)^c = E_1^c \cap E_2^c \cap \ldots \cap E_k^c$

Notation Practice:
Explain what is wrong with each of the following:

a) $\Pr(A) \cup \Pr(B)$

b) $[\Pr(A)]^c$

c) Pr(1-A)

d) Pr(A+B)