**STAT 325 – Handout 20**
**Poisson Process (7.2, 7.3)**

- Let $N(t)$ denote the number of occurrences of an event in the interval $[0, t]$. Then $N(t)$ is a **Poisson process** if three conditions hold:
  - Events in non-overlapping intervals occur independently.
  - The probability distribution of number of events depends only on the length of the interval, not on the starting or ending point of the interval.
  - The probability of $x$ events in an interval of length $t$ is:

$$\Pr[N(t) = x] = e^{-\lambda t} \frac{(\lambda t)^x}{x!}$$

    - So $N(t)$ has a Poisson distribution with mean $\lambda t$
  - The parameter $\lambda$ is the *rate*, or expected number, of events per unit of time.

**Example 20-1: ATM machine**

Suppose that the number of visitors to a particular ATM machine on campus follows a Poisson process with rate $\lambda = .2$ per minute.

a) Determine the probability that exactly 5 people visit the ATM machine between 11:00 and 11:30. [*Hint*: First determine the probability distribution of the random variable $N(30)$.]

b) Determine the probability that exactly 5 people visit the ATM machine between 2:45 and 3:15.

c) Determine the probability that at least 5 people visit the ATM in any half hour.

d) Determine the probability that at least 10 minutes elapse between visits to the ATM. [*Hint*: Express this probability in terms of the number of visits to the ATM in that time period.]

e) For any positive value *t*, determine the probability that at least *t* minutes elapse between visits to the ATM.

f) For any positive value *t*, determine the probability that *t* or fewer minutes elapse between visits to the ATM.

Let the random variable T represent the time between visits to the ATM.

g) Determine the cdf and pdf of T. If possible, identify the probability distribution of T by name.

h) More generally (for the moment), let the rate of visits to the ATM be $\lambda$, and again let T represent the time between visits. Determine the probability distribution of T.

- Let the random variable T represent the time between consecutive events in a Poisson process with rate $\lambda$. Then T has an *exponential* distribution with parameter $\lambda$.
  - So, $E(T) = 1/\lambda$, $Var(T) = 1/\lambda^2$, $SD(T) = 1/\lambda$
  - Inter-arrival times in a Poisson process are *independent* exponential distributions.

i) Determine the expected value of the waiting time between visits to the ATM.

j) Suppose that you observe the ATM until you see 5 visits. Determine the probability that you will have to wait for more than 30 minutes. [*Hint*: Re-express this probability in terms of the number of visits that occur in 30 minutes.]

k) More generally, let $\lambda$ be the rate of the Poisson process, and determine the probability that you will have to wait for more than *t* minutes in order to observe *k* visits.

- Let W represent the waiting time for the $k^{\text{th}}$ event in a Poisson process with rate $\lambda$.
  - The cdf of W is: $F(w) = 1 - \sum_{j=0}^{k-1} e^{-\lambda w} \frac{(\lambda w)^j}{j!}$ for $w > 0$
  - The pdf of W is: $f(w) = \frac{\lambda^k w^{k-1}}{(k-1)!} e^{-\lambda w}$
  - W is said to have an **Erlang distribution** with parameters *k* and $\lambda$.
    - W can be expressed as the sum of *k* independent exponential distributions.
  - $E(W) = k/\lambda$, $Var(W) = k/\lambda^2$

l) Determine the expected value and SD of your waiting time to observe 5 visits to the ATM.