STAT 325 – Handout 22
“Choosing the Best”: Decision Making under Uncertainty
(My Favorite Problem!)

Problem:

Your supervisor has assigned you the task of hiring a new employee subject to these rules:

1. A total of *n* candidates have applied for the position.
2. The candidates arrive in random order.
3. You interview candidates one at a time.
4. You can rank the candidates that you have interviewed from best to worst, but you have no *a priori* knowledge about the quality of the candidates.
5. Once you have interviewed a candidate, you must decide *immediately* whether to hire him/her. If you decide to hire, the process ends. If you opt not to hire, the process continues but that candidate can no longer be considered.
6. Your supervisor will be satisfied only if you hire the *best* candidate. Hiring the second best candidate is no better than hiring the very worst.

Predictions:

Before we start, take a guess concerning the optimal probability of success for the numbers of applicants listed in the table. (We will determine the actual probabilities in the next hour.)

	applicants:	3	12	500	6,822,961,365
optimal probability of success	guess:				
	actual:				

Strategy:

Start by considering possible strategies in cases with few applicants.

1. For $n = 1$, there is only one possible interview order: A: 1

2. For $n = 2$, there are two possible interview orders: A: 12 B: 21

3. For $n = 3$, there are six possible interview orders:

   A: 123 B: 132 C: 213
   D: 231 E: 312 F: 321

4. For $n = 4$, there are twenty-four possible interview orders:

   A: 1234 B: 1243 C: 1324 D: 1342
   E: 1423 F: 1432 G: 2134 H: 2143
   I: 2314 J: 2341 K: 2413 L: 2431
   M: 3124 N: 3142 O: 3214 P: 3241
   Q: 3412 R: 3421 S: 4123 T: 4132
   U: 4213 V: 4231 W: 4312 X: 4321

These relatively simple cases suggest a general form for the optimal strategy:

Analysis:
Let *n* be the number of applicants and *r* be the position of first "contender." Since the best candidate is equally likely to be in any of the *n* possible positions, the probability of choosing the best is simply the average of the conditional probabilities of choosing the best given each possible position of the best candidate:

- If the best candidate is in one of the first *r*-1 positions, the success probability is:

- If the best candidate is in position *r*, the probability of choosing her/him is:

- If the best candidate is past position *r* and in position *i*, the success probability is:

Thus, the overall probability of choosing the best can be expressed as:

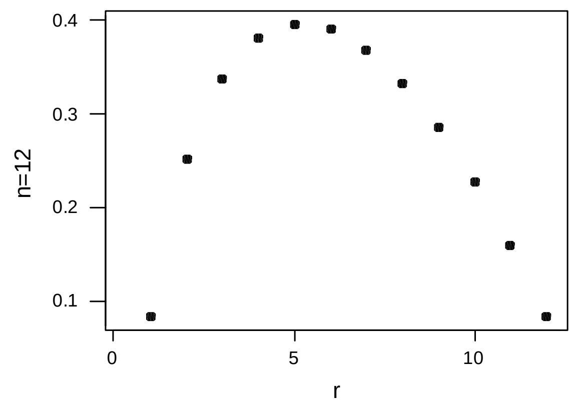
For a given number of applicants *n*, we can determine this probability for each possible value of *r* and then choose *r* to maximize this probability.

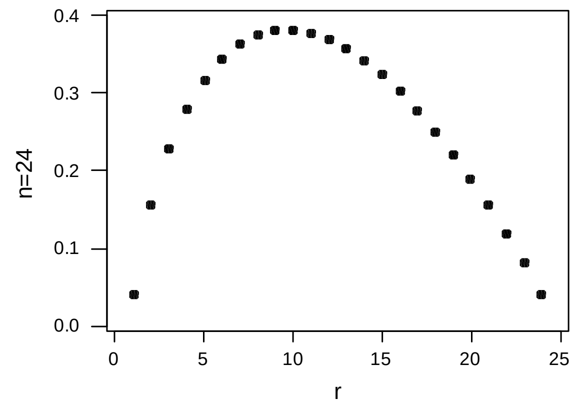
r \ *n*	1	2	3	4	5	6	7	8
1	1	0.5	0.3333	0.25	0.2	0.1667	0.1429	0.125
2		0.5	**0.5**	**0.4583**	0.4167	0.3806	0.35	0.3241
3			0.3333	0.4167	**0.4333**	**0.4278**	**0.4143**	0.3982
4				0.25	0.35	0.3917	0.4071	**0.4098**
5					0.2	0.3	0.3524	0.3798
6						0.1667	0.2619	0.3185
7							0.1429	0.2321
8								0.125

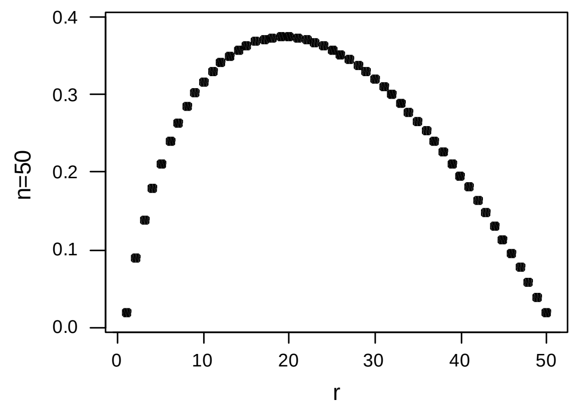
r \ *n*	9	10	11	12	13	14	15	16
1	0.1111	0.1	0.0909	0.0833	0.0769	0.0714	0.0667	0.0625
2	0.302	0.2829	0.2663	0.2517	0.2387	0.2272	0.2168	0.2074
3	0.3817	0.3658	0.3507	0.3366	0.3236	0.3114	0.3002	0.2898
4	**0.406**	**0.3987**	0.3897	0.38	0.37	0.36	0.3503	0.3409
5	0.3931	0.3983	**0.3984**	**0.3955**	0.3907	0.3848	0.3782	0.3712
6	0.3525	0.3728	0.3844	0.3902	**0.3923**	**0.3917**	**0.3894**	0.3859
7	0.2897	0.3274	0.3522	0.3683	0.3784	0.3843	0.3873	**0.3881**
8	0.2083	0.2653	0.3048	0.3324	0.3517	0.3651	0.3741	0.3799
9	0.1111	0.1889	0.2444	0.2847	0.3141	0.3356	0.3513	0.3627
10		0.1	0.1727	0.2265	0.2668	0.2972	0.3202	0.3377
11			0.0909	0.1591	0.211	0.2508	0.2817	0.3058
12				0.0833	0.1474	0.1973	0.2366	0.2676
13					0.0769	0.1374	0.1853	0.2238
14						0.0714	0.1286	0.1747
15							0.0667	0.1208
16								0.0625

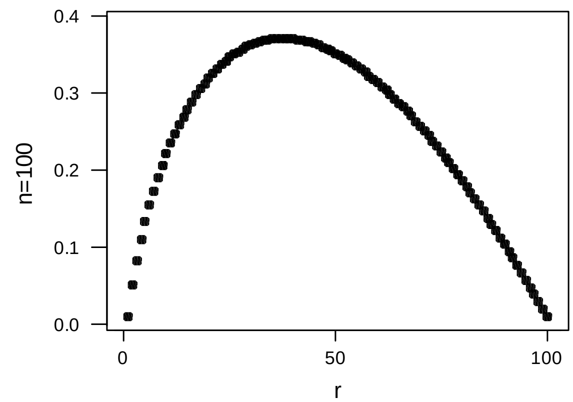
r \ *n*	17	18	19	20	21	22	23	24
1	0.0588	0.0556	0.0526	0.05	0.0476	0.0455	0.0435	0.0417
2	0.1989	0.1911	0.184	0.1774	0.1713	0.1657	0.1605	0.1556
3	0.2801	0.2711	0.2626	0.2548	0.2474	0.2405	0.234	0.2279
4	0.3319	0.3233	0.315	0.3072	0.2997	0.2925	0.2858	0.2793
5	0.3641	0.3569	0.3498	0.3429	0.3361	0.3295	0.323	0.3168
6	0.3816	0.3767	0.3715	0.3661	0.3606	0.355	0.3495	0.3439
7	**0.3873**	**0.3854**	0.3827	0.3793	0.3755	0.3715	0.3672	0.3627
8	0.3832	0.3848	**0.385**	**0.3842**	0.3826	0.3803	0.3776	0.3746
9	0.3708	0.3763	0.3799	0.382	**0.3828**	**0.3827**	**0.3819**	0.3805
10	0.3509	0.3608	0.3682	0.3734	0.3771	0.3794	0.3807	**0.3812**
11	0.3246	0.3392	0.3506	0.3594	0.3661	0.3711	0.3747	0.3772
12	0.2923	0.312	0.3278	0.3403	0.3503	0.3582	0.3644	0.3691
13	0.2547	0.2798	0.3001	0.3167	0.3302	0.3412	0.3501	0.3572
14	0.2122	0.2429	0.2681	0.2889	0.3061	0.3204	0.3321	0.3418
15	0.1652	0.2018	0.2321	0.2573	0.2784	0.2961	0.3108	0.3233
16	0.114	0.1567	0.1923	0.2221	0.2473	0.2685	0.2865	0.3017
17	0.0588	0.1078	0.149	0.1836	0.213	0.2379	0.2592	0.2774
18		0.0556	0.1023	0.142	0.1757	0.2045	0.2292	0.2504
19			0.0526	0.0974	0.1356	0.1684	0.1966	0.2211
20				0.05	0.0929	0.1298	0.1617	0.1894
21					0.0476	0.0887	0.1244	0.1555
22						0.0455	0.085	0.1195
23							0.0435	0.0815
24								0.0417

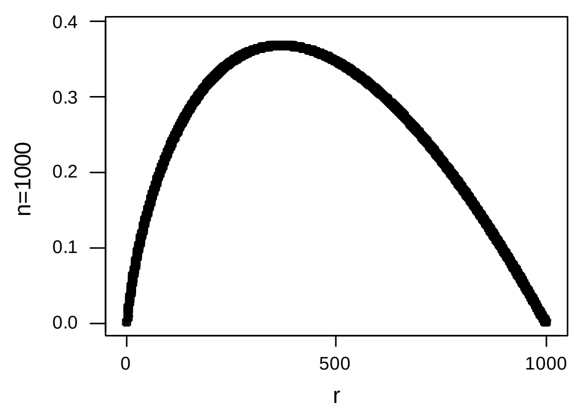
Some pictures:

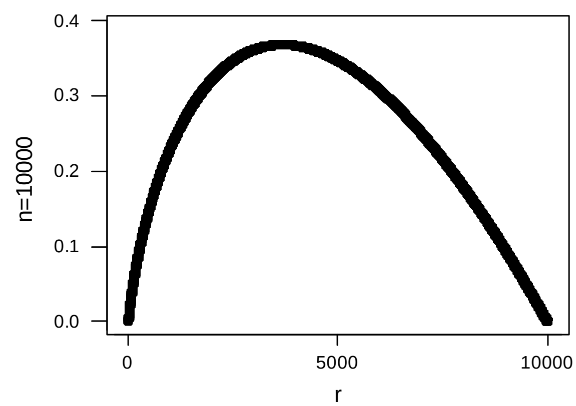


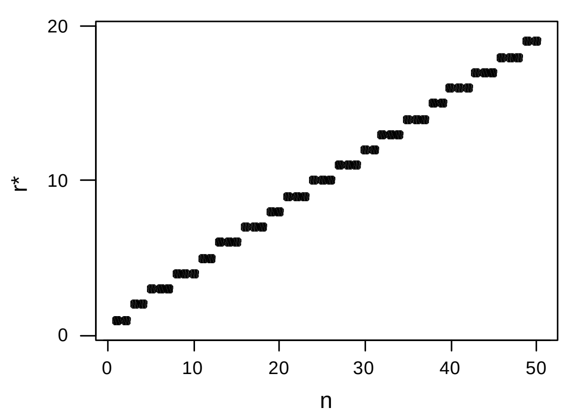


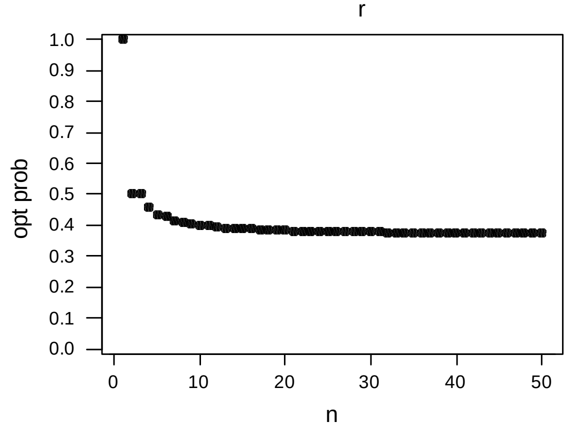










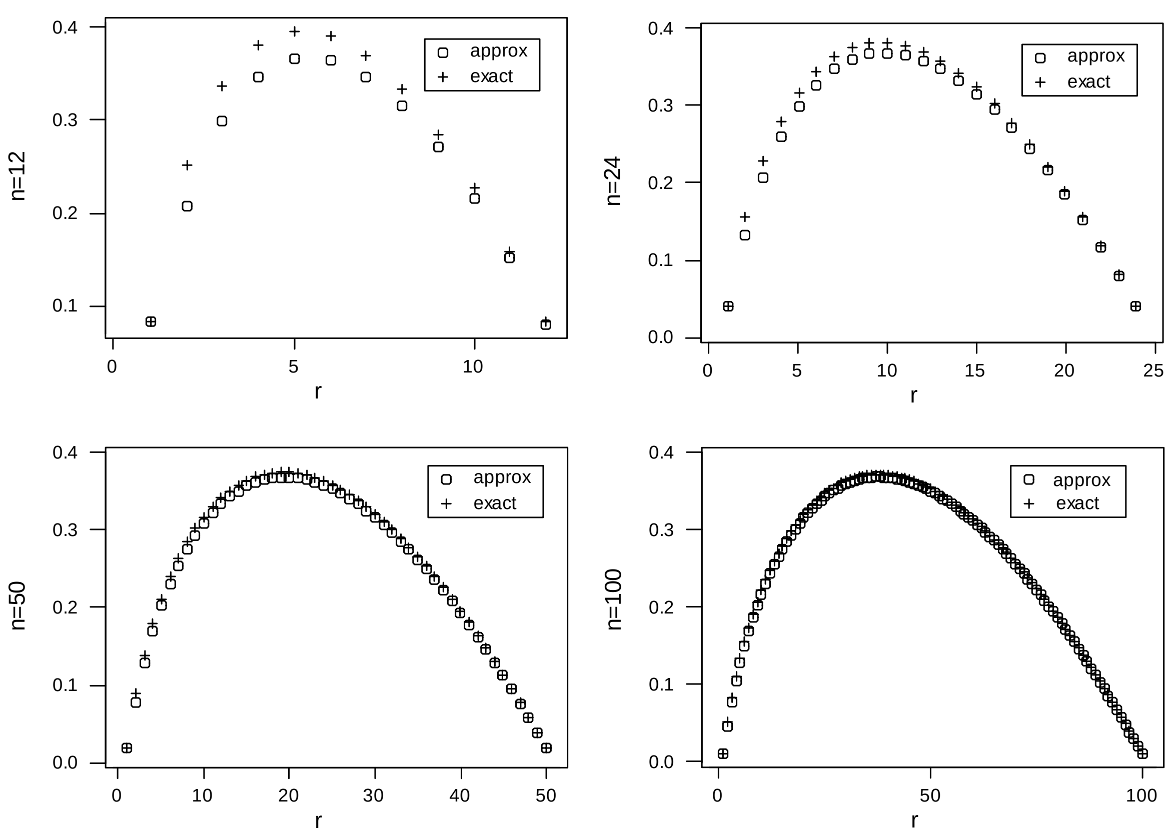


Some practice:
Apply the optimal strategy on the following simulated orderings of 12 applicants:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1	4	8	7	4	7	4	2	9	12	1	5	3	12	1	5	3	1	12	3	6	3	6	8	5	7
2	12	1	5	3	12	1	5	3	11	11	10	7	8	5	11	11	10	7	1	12	4	8	7	4	3
3	1	12	3	6	3	6	8	5	7	4	2	9	6	10	12	1	9	2	4	8	10	7	1	12	10
4	5	3	9	2	9	2	4	8	1	12	3	6	3	6	8	5	2	9	6	10	2	9	6	10	4
5	6	10	12	1	1	12	3	6	8	5	11	11	5	3	9	2	12	1	5	3	12	1	5	3	11
6	7	4	2	9	6	10	12	1	5	3	9	2	9	2	4	8	6	10	12	1	1	12	3	6	8
7	11	11	10	7	8	5	11	11	6	10	12	1	1	12	3	6	4	8	7	4	7	4	2	9	12
8	3	6	8	5	11	11	10	7	2	9	6	10	2	9	6	10	3	6	8	5	11	11	10	7	1
9	8	5	11	11	5	3	9	2	9	2	4	8	10	7	1	12	7	4	2	9	6	10	12	1	5
10	10	7	1	12	4	8	7	4	3	6	8	5	11	11	10	7	8	5	11	11	5	3	9	2	9
11	2	9	6	10	2	9	6	10	4	8	7	4	7	4	2	9	11	11	10	7	8	5	11	11	6
12	9	2	4	8	10	7	1	12	10	7	1	12	4	8	7	4	5	3	9	2	9	2	4	8	2

Approximate analysis:
For large values of *n*:

and thus:



Asymptotic analysis:
Setting $f'(r)$ equal to 0 and solving for *r* gives that this probability is maximized for:

For large values of *n*, the optimal probability is then:

Thus, for large values of *n*, the optimal procedure is:

With this strategy, you have a probability of *choosing the best* roughly equal to:

Extensions:

1. Number game

2. Finding a soul-mate

References:

DeGroot, M.H. (1970), *Optimal Statistical Decisions*, McGraw-Hill, New York.
DeGroot, M.H. (1986), *Probability and Statistics* (2nd ed.), Addison-Wesley, Reading MA.
Ferguson, T.S. (1989), "Who Solved the Secretary Problem?", *Statistical Science*, 4, 282-289.