STAT 325 – Handout 4
Counting Techniques (1.4)

Recall that when the outcomes of a random process are considered to be equally likely, for example when an object is chosen at random from a population, the probability of an event is the number of outcomes in the event divided by the total number of outcomes in the sample space. Therefore, it's very useful to be able to count efficiently.

Multiplication rule of counting: Suppose that the first stage of a process can be completed in any one of n_1 ways. Further, suppose that for each way of completing the first stage, the second stage can be completed in any one of n_2 ways. Then the two-stage process can be completed in any one of $n_1 \times n_2$ ways. This rule extends naturally to a k-stage process, which can then be completed in any one of $n_1 \times n_2 \times \ldots \times n_k$ ways.

Example 4-1: Ice Cream Sundaes
a) How many ice cream sundaes can you choose from if you are presented with three flavors of ice cream (vanilla, chocolate, strawberry) and four toppings (hot fudge, marshmallow, butterscotch, peanut butter)? Also list them.

b) What if you can also choose whether to have nuts or not, and whipped cream or not, and a cherry or not?

A result that follows directly from the multiplication rule of counting is:

- The number of ordered arrangements of n items is $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$.
    - Also written as $n!$ (pronounced "n factorial").
    - By definition, $0! = 1$

Example 4-2: Boy/Girl Seating
Suppose that 4 men and 4 women are going to sit in a row.
a) In how many ways can they sit?

b) In how many ways can they sit so nobody sits next to someone of the same gender?

c) If they choose their seats completely at random, what is the probability that nobody sits next to someone of the same gender?

Now suppose that there are *n* men and *n* women who are going to sit in a row.

d) Intuitively, how would you expect this probability to change as *n* increases? Explain.

e) Determine this probability. Is your intuition confirmed? Explain.

Example 4-3: Random Babies (cont.)
Recall that we have simulated the process of giving four babies back to their mothers at random.

a) How many distinct ways are there to return four babies to four mothers? In other words, how many outcomes are in the sample space?

b) Write out the sample space of all possible outcomes.

c) Count how many of these outcomes produce at least one mother with the correct baby.

d) Report the (exact) probability that at least one mother receives the correct baby. Is this close to our approximations from simulation?

Another result that follows directly from the multiplication rule of counting is:

- The number of ordered arrangements of *k* items, selected from a set of *n* distinct items, is $n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$.
  - Also expressed as $n!/(n-k)!$
  - Known as the number of **permutations** of *n* distinct items taken *k* at a time

Example 4-4: Birthday Problem
A classic problem asks about the probability that at least two people in a group of n share the same birthday? Let's ignore February 29 and assume that the other 365 days are all equally likely to be the birthday of a randomly selected person. Let's begin with n = 25, the number of students in this class.

a) Make a guess for the probability that at least two people in a group of 25 share the same birthday.

b) How many different assignments of birthdays to these 25 people are possible? Is this the numerator or denominator of the probability?

c) Calculate the numerator of the probability by counting how many birthday assignments are possible so every person has a different birthday.

d) Calculate the probability that every person has a different birthday. Then use this to find the probability that at least two people share the same birthday.

e) Give a general expression for the probability of shared birthdays in a group of n people.

f) Use R to calculate and graph this probability for all values of n from 1 to 100.

Example 4-5: Choosing Officers
a) Suppose that a president and a secretary are to be selected from a group of 6 people. How many choices are possible?

b) Now suppose that two people are to be selected from group of 6 people, where order does not matter. Would there be more or fewer possibilities than in a)? By what factor?

- The number of ways to choose *k* items from a group of *n* distinct items, where order does not matter, is $\frac{n!}{k!(n-k)!}$.
  - Also denoted by $\binom{n}{k}$, pronounced "*n* choose *k*"
  - Known as the number of **combinations** of *n* items taken *k* at a time
  - Also known as a binomial coefficient
  - Often displayed in a diagram known as Pascal's triangle

Example 4-6: College Committee Formation
Recall the question from HW1 about choosing a subgroup of 2 people from a college committee consisting of 4 men (Abe, Ben, Chuck, Don) and 2 women (Ellen, Frannie).

a) How many ways are there to choose the subgroup?

b) How many ways are there to choose a subgroup containing two women?

c) If the subgroup of size 2 is selected at random, what is the probability that it contains both women?

d) If a subgroup of size 3 is selected at random, what is the probability that it contains both women? [*Hint*: Be careful with the numerator. Be sure to include the number of ways of selecting the one man for the subgroup.]

e) Now suppose that the committee consists of *M* men and 2 women. What is the probability that a randomly selected subgroup of 3 committee members would include 2 women?

f) Is this an increasing or decreasing function of *M*? Explain why this makes sense?

Example 4-7: Gender Discrimination?

In a recent study, researchers presented bank managers with identical personnel files and asked them whether the candidate was worthy of promotion. Half of the managers were randomly assigned to receive a file with a male name attached, and the other half were randomly assigned to receive a file with a female name attached. The researchers suspected that the managers would be more likely to recommend promotion for candidates with male names. The resulting experimental data are summarized in the following two-way table:

	Male name	Female name	Total
Promotion	21	14	35
No promotion	3	10	13
Total	24	24	48

a) Suppose that there's really no discrimination of any kind at work here, so the 35 managers who voted for promotion would have done so regardless of the name on the file, as would the 13 managers who voted against promotion. If we then randomly select 24 of the 48 files to be in the "male name" group, what is the probability that we would obtain 21 or more of the files judged to be worthy of promotion?

b) Is this probability small enough to cast strong doubt on the assumption of no discrimination at work here? Explain.