

STAT 325 – Handout 5
Conditional Probability (1.5)

One of the most important concepts in the study of randomness is *conditional probability*, which allows for updating uncertainty in light of partial information about how the random process turned out.

Example 5-1: Top 100 Films (cont.)

Recall the following table classifying “top 100” American films by whether Allan has seen the film and by whether Beth has seen the film:

	Beth yes	Beth no	Total
Allan yes	42	6	48
Allan no	17	35	52
Total	59	41	100

If one of these 100 films is selected at random, $\Pr(A) = .48$ and $\Pr(B) = .59$, where A represents the event that Allan has seen the film and similarly for B with Beth.

Now suppose that one of these 100 films is selected at random, and you are told that Beth has seen the film selected.

a) Does this (partial) information about the film change your belief about how likely it is that Allan has seen the film? Explain.

b) According to the table, among the 59 films that Beth has seen, what proportion has Allan seen?

- For any two events with $\Pr(B) > 0$, the **conditional probability** of A given B, denoted as $\Pr(A|B)$, is defined as: $\Pr(A|B) = \Pr(A \cap B) / \Pr(B)$.

c) Use this definition to calculate $\Pr(A|B)$. Does the calculation agree with your answer to b)?

d) Use this definition to calculate $\Pr(A^c|B)$. How does this compare to $\Pr(A|B)$? Does this make sense?

- $\Pr(A^c|B) = 1 - \Pr(A|B)$
 - Make sure that you condition on the same event before using this rule.

e) Calculate and interpret the value of $\Pr(B|A)$.

f) Calculate and interpret the value of $\Pr(A \cap B | A \cup B)$.

Example 5-2: Male Senators

a) Do you think $\Pr(E | F)$ and $\Pr(F | E)$ always have to be the same? Similar? Or can they be very different?

Suppose that we select an American citizen at random. Define the events $M = \{\text{the American citizen is male}\}$ and $E = \{\text{the American citizen is a U.S. Senator}\}$.

b) Make a reasonable guess for the value of the conditional probability $\Pr(M | E)$.

c) Make a reasonable guess for the value of the conditional probability $\Pr(E | M)$.

d) Are these conditional probabilities equal? Similar? Vastly different?

Example 5-3: Children

Suppose that a couple has two children, with all four gender possibilities $\{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$ equally likely.

a) What is the probability that both children are boys?

b) If you learn that the older child is a boy, what is the conditional probability that both children are boys?

c) If you learn that at least one child is a boy, what is the conditional probability that both children are boys?

d) If you observe the couple walking down the street with one of their children, and you see that the child is a boy, what is the conditional probability that both children are boys?

Example 5-4: Polya's Urn Scheme

Suppose that an urn contains 4 red balls and 4 green balls. You select one ball at random and then replace it with 2 more balls of the same color. This process continues, with each selected ball being replaced with 2 more of the same color. Let R_i denote the event that the i^{th} selection is a red ball, and similarly for G_i .

a) What is the conditional probability that the second ball selected is red, given that the first is red? Be sure to express this probability with appropriate notation.

b) Use set notation to denote the event that the first 2 balls selected are both red.

c) Perform some algebra on the definition of conditional probability to derive an expression for the probability of the intersection between two events.

- **Multiplication rule:** $\Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$, as long as $\Pr(B) > 0$.
 - Also $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$, as long as $\Pr(A) > 0$.

d) Use the multiplication rule to determine the probability that the first 2 balls selected are both red.

e) Produce a *probability tree* to determine the probabilities for all possible intersections involving 2 selections.

- **Multiplication rule** (more general): $\Pr(E_1 \cap E_2 \cap \dots \cap E_k) = \Pr(E_1) \times \Pr(E_2 | E_1) \times \Pr(E_3 | E_1 \cap E_2) \times \dots \times \Pr(E_k | E_1 \cap E_2 \cap \dots \cap E_{k-1})$.

f) Determine the probability that the first 4 selections would all result in the same color.

Example 5-5: Ballot Problem

Suppose that candidate A receives 4 votes in an election, and the only other candidate (B) receives 2 votes. Suppose that the 6 votes are counted in random order, with the current vote totals for each candidate tallied after each vote is counted. What is the probability that candidate A never trails candidate B as the count proceeds?

a) How many outcomes are in the sample space of this random process?

b) Do you want to solve this problem by listing the sample space of possible outcomes? [*Hint:* The correct answer contains 2 letters.]

c) Let E denote the event that candidate A never trails candidate B as the count proceeds. Describe the event E^c .

Now let A_i mean that candidate A receives the i^{th} vote to be counted, and similarly define B_i .

d) There are only two ways that the event E^c can occur. What's the simpler way, in terms of A_i 's and/or B_i 's? Calculate this probability.

e) What is the more complicated way that the event E^c can occur, in terms of A_i 's and/or B_i 's? Use the multiplication rule to calculate this probability.

f) Is it legitimate to just add these two probabilities to determine $\Pr(E^c)$? Explain why or why not.

g) Report the probability of the event E.

You can imagine that these probabilities become very hard to determine analytically with larger numbers of ballots. But these probabilities can be well approximated by computer simulations.

Example 5-6: Polya's Urn Scheme (cont.)

Reconsider Polya's urn scheme, applied to selecting a total of two balls.

a) Use the probability tree to determine the probability that the 2nd ball selected is red (regardless of the color of the 1st ball).

- The events B_1, B_2, \dots, B_k form a **partition** of the sample space S if:
 - The B_i 's are mutually exclusive.
 - The union of the B_i 's equals S.
- **Law of Total Probability (LTP):** If A is any event and B_1, B_2, \dots, B_k form a partition of the sample space S, then $\Pr(A) = \sum_{i=1}^k \Pr(A | B_i) \Pr(B_i)$.

b) Show how the LTP applies to the previous question.

Example 5-7: Humans vs. Rats

You will be shown a sequence of randomly generated red/green displays. Then you will be asked to predict the color for the next set of randomly generated displays.

a) Record your predictions here.

b) You will be told the strategy that most humans use for this guessing game. Use a probability tree and/or the Law of Total Probability to determine the probability of a correct guess.

c) You will be told how laboratory rats play this game, when the color shown indicates which door a food pellet will emerge from. Determine the probability that a rat makes a correct guess.