STAT 325 Introduction to Probability Models Spring 2010

HW11 due Thur Apr 29

Topics: Geometric, negative binomial, hypergeometric, Poisson random variables

1. Suppose that I generate two English letters at random, with replacement.

a) How many ways are there to choose this pair of letters?

$26 \times 26 = 676$

According to the website http://phrontistery.info/scrabble3.html, there are 96 legitimate two-letter English words.

b) Use this information, and your answer to a), to determine the probability that a randomly chosen pair of letters creates a legitimate English word.

This probability is $96/676 = 24/169 \approx .142$.

Now suppose that you generate random two-letter pairs, with replacement, over and over until you obtain a legitimate English word for the first time. Let the random variable X represent the number of two-letter pairs that you must generate.

c) What probability distribution does X have? Specify its parameter value(s) as well as its name.

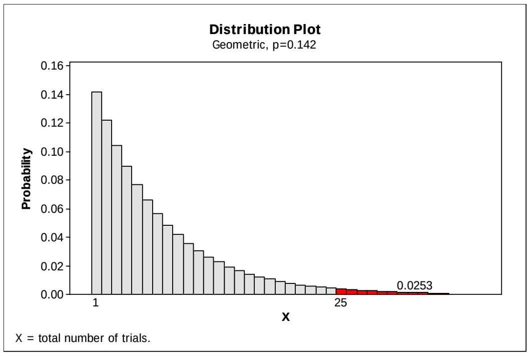
$X \sim \text{Geometric}(p \approx .142)$

d) Determine the expected value and standard deviation of the number of pairs that you must generate.

$E(X) = 1/p \approx 1/.142 \approx 7.042$
$Var(X) = (1-p)/(p^2) \approx .858/(.142^2) \approx 42.543$
$SD(X) = \text{sqrt}[Var(X)] \approx 6.523$

e) Determine the probability that you need to generate at least 25 pairs in order to find a legitimate English word. (Feel free to use Minitab or R or another software program.)

$\Pr(X \geq 25) = \sum_{x=25}^{\infty} (.142)(.858)^{x-1} \approx .0253$, as shown in the Minitab graph:



2. Our class consists of 12 STAT majors and 14 other majors. Suppose that I select 3 students at random, without replacement. Let M be the number of STAT majors selected.

a) What probability distribution does M have? Specify its parameter value(s) as well as its name.

M ~ Hypergeometric($N = 26$, $r = 12$, $n = 3$)

b) Determine the expected value and standard deviation of M.

$E(M) = nr/N = 3(12/26) \approx 1.385$
$Var(M) = nr/N(1-r/N)[(N-n)/(N-1)] = 3(12/26)(14/26)(23/25) \approx 0.686$
$SD(M) = \text{sqrt}[Var(M)] \approx 0.828$

c) Determine the probability that STAT majors outnumber other majors in the sample.

$$\Pr(M \geq 2) = \Pr(M = 2) + \Pr(M = 3) = \frac{\binom{12}{2}\binom{14}{1}}{\binom{26}{3}} + \frac{\binom{12}{3}\binom{14}{0}}{\binom{26}{3}} = .44$$

Now suppose that I select 3 students at random, *with* replacement.

d) Re-answer parts a)-c) for this scenario.

Now M ~ Binomial($n = 3$, $p = 12/26 \approx .462$)

$E(M) = np = 3(12/26) \approx 1.385$
$Var(M) = np(1-p) = 3(12/26)(14/26) \approx 0.746$
$SD(M) = \text{sqrt}[Var(M)] \approx 0.863$

$$\Pr(M \geq 2) = \Pr(M = 2) + \Pr(M = 3) = \binom{3}{2}(.462)^2(.538)^1 + \binom{3}{3}(.462)^3(.538)^0 \approx .442$$

e) Comment on how your answers to b) and c) differ between these two scenarios.

The expected values are the same. The variance and SD are slightly larger when the sampling is with replacement. The probabilities are quite similar, a bit larger when the sampling is with replacement.

3. Suppose that the number of colds that an individual suffers in one year is modeled as a Poisson random variable with parameter $\mu = 0.6$.

a) Determine the probability that an individual has at least one cold in a year.

Let $X \sim \text{Poisson}(\mu = 0.6)$. Then $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - e^{-.6}(.6)^0/0! = 1 - e^{-.6} \approx 1 - .549 = .451$

b) Determine the probability that an individual has at least two colds in a year.

$\Pr(X \geq 2) = 1 - \Pr(X = 0) - \Pr(X = 1) = 1 - e^{-.6}(.6)^0/0! - e^{-.6}(.6)^1/1! \approx 1 - 1.6e^{-.6} = 1 - .878 = .122$

c) Determine the conditional probability that an individual has exactly one cold given that he/she has at least one cold.

$\Pr(X = 1 \mid X \geq 1) = \Pr[(X = 1) \cap (X \geq 1)] / \Pr(X \geq 1) = \Pr(X = 1) / \Pr(X \geq 1) \approx .329/.451 \approx .729$

Now suppose that the manufacturer of a zinc tablet claims that individuals who take such a tablet once a day reduce the Poisson parameter to $\mu = 0.3$.

d) Determine the probability that an individual taking this zinc tablet has at least one cold in a year.

Let $Y \sim \text{Poisson}(\mu = 0.3)$. Then $\Pr(Y \geq 1) = 1 - \Pr(Y = 0) = 1 - e^{-.3}(.3)^0/0! = 1 - e^{-.3} \approx 1 - .741 = .259$

e) If the manufacturer's claim is true, how many times more likely is an individual to remain cold-free if he/she takes the zinc tablet than if he/she does not take the zinc tablet?

$\Pr(Y = 0) / \Pr(X = 0) = [e^{-.3}(.3)^0/0!] / [e^{-.6}(.6)^0/0!] = e^{-.3}/e^{-.6} = e^{.3} \approx 1.35.$