

## STAT 325 Introduction to Probability Models Spring 2010

### HW13 due Wed May 5

Topics:  $k$ -step transition probabilities, regular Markov chain, steady-state probabilities

1. A printer is always in one of two states: printing or idle. Suppose that the printer's status is noted every minute to see if it is printing or idle. This process follows a Markov chain with the following 1-step transition probability matrix:

	Printing	Idle
Printing	.4	.6
Idle	.2	.8

a) Suppose that the printer is idle at 12 noon. Determine the probability that it will also be idle at 12:02. Show how you perform this calculation without using matrix algebra.

Let  $I_2$  denote the event that the printer is idle at 12:02. By the law of total probability,  $\Pr(I_2) = \Pr(I_2 | I_1) \times \Pr(I_1) + \Pr(I_2 | P_1) \times \Pr(P_1) = (.8)(.8) + (.6)(.2) = .76$ .

b) Suppose that the printer is idle at 12 noon. Determine the probability that it will also be idle at both 12:01 and 12:02. [Hint: Express this event with set operations, and use a rule that we learned early in the course.]

By the multiplication rule,  $\Pr(I_1 \cap I_2) = \Pr(I_1) \times \Pr(I_2 | I_1) = (.8)(.8) = .64$ .

c) Suppose that the printer is idle at 12 noon. Determine the probability that it will also be idle at 12:05. Indicate how you use matrix operations to calculate this, and feel free to use R or another software program.

This probability is the bottom right entry in the matrix  $P^5$ , where  $P$  is the 1-step transition probability matrix given above. It turns out that  $P^5 = \begin{pmatrix} .25024 & .74976 \\ .24992 & .75008 \end{pmatrix}$ , so the probability that the printer is idle at 12:05 is .75008.

d) Set up a system of equations to be solved in order to determine the steady-state probability vector for this Markov chain. Then use substitution to solve that system of equations.

Let  $\pi_1$  denote the steady-state probability that the printer is printing and let  $\pi_2$  denote the steady-state probability that the printer is idle. The system of equations comes from solving  $\pi P = \pi$  and  $\pi_1 + \pi_2 = 1$ . This gives:

$$.4 \pi_1 + .2 \pi_2 = \pi_1$$

$$.6 \pi_1 + .8 \pi_2 = \pi_2$$

$$\pi_1 + \pi_2 = 1$$

The last equation gives that  $\pi_2 = 1 - \pi_1$ . Substituting this into the first equation gives:

$$.4 \pi_1 + .2(1 - \pi_1) = \pi_1, \text{ so } .8 \pi_1 = .2, \text{ so } \pi_1 = .25 \text{ and } \pi_2 = .75.$$

(Note that this is consistent with the two rows of the matrix  $P^5$  above.)

e) Interpret what these steady-state probabilities tell you about this printer.

In the long run, the printer is printing 25% of the time and is idle 75% of the time.

2. Suppose that Armando has \$1 and Blanca has \$3. They play a game based on flipping a fair coin. If the coin lands heads, then Armando pays Blanca \$1, and if the coin lands tails, then Blanca pays Armando \$1. They continue playing until one of them has all \$4.

a) Let the states of this stochastic process be how many dollars Armando has. Determine the 1-step transition probability matrix for this Markov chain.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

b) Indicate how to use matrix operations to determine the probability distribution for how many dollars Armando has after 2 coin flips. Then do the matrix calculations to determine this probability distribution. (Feel free to use R, or another software program, to do the matrix calculations.) Also calculate the expected value of how many dollars Armando has after 2 coin flips.

This probability distribution is the second row (which corresponds to starting with \$1) of the matrix  $P^2$ .

$$\text{It turns out that } P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .5 & .25 & 0 & .25 & 0 \\ .25 & 0 & .5 & 0 & .25 \\ 0 & .25 & 0 & .25 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ so the probability distribution of Armando's}$$

fortune after 2 flips is:

$x$	0	1	2	3	4
$p(x)$	.5	.25	0	.25	0

The expected value is:  $E(X) = 0(.5) + 1(.25) + 3(.25) = \$1$ .

c) Repeat b) for Armando's fortune after 16 coin flips.

The second row of the matrix  $P^{16}$  reveals that the probability distribution of Armando's fortune (to 3 decimal places) after 16 flips is:

$x$	0	1	2	3	4
$p(x)$	.748	.002	0	.002	.248

The expected value is:  $E(X) = 0(.748) + 1(.002) + 2(0) + 3(.002) + 4(.248) = \$1$ .

d) What do these calculations suggest will happen to this probability distribution (of how many dollars Armando has) as the game continues?

It appears that Armando has probability .75 of ending up with \$0 and probability .25 of ending up with \$4 dollars. The expected value of his fortune remains at \$1 throughout, not surprisingly since this is a fair (50/50) game on each flip.

3. Reconsider the previous exercise, but now suppose that they roll a fair six-sided die to determine how the \$4 will be distributed at the start of the game. If it lands on 1 or 2 or 3, then Armando starts with \$1 and Blanca with \$3. If it lands on 4 or 5, then they both start with \$2. If it lands on 6, then Armando starts with \$3 and Blanca with \$1.

a) Determine the probability distribution for how many dollars Armando has after 1 coin flip. Solve this without using matrix algebra, and show the steps of your solution.

Let  $Y$  represent Armando's fortune after 1 coin flip, let  $S_i$  be the event that he starts with  $\$i$ , and let  $H_i$  be the event that flip  $i$  lands on heads.

$$\Pr(Y = 0) = \Pr(S_1 \cap H_1) = \Pr(S_1) \times \Pr(H_1) \text{ because these are independent} = (3/6)(1/2) = 1/4 = .25$$

$$\Pr(Y = 1) = \Pr(S_2 \cap H_1) = \Pr(S_2) \times \Pr(H_1) = (2/6)(1/2) = 1/6 \approx .167$$

$$\Pr(Y = 2) = \Pr(S_3 \cap H_1) + \Pr(S_1 \cap T_1) = \Pr(S_3) \times \Pr(H_1) + \Pr(S_1) \times \Pr(T_1) = (1/6)(1/2) + (3/6)(1/2) = 1/3 \approx .333$$

$$\Pr(Y = 3) = \Pr(S_2 \cap T_1) = \Pr(S_2) \times \Pr(T_1) = (2/6)(1/2) = 1/6 \approx .167$$

$$\Pr(Y = 4) = \Pr(S_3 \cap T_1) = \Pr(S_3) \times \Pr(T_1) = (1/6)(1/2) = 1/12 \approx .083$$

b) Based on your answer to a), what is the probability that the game ends after just 1 coin flip?

$$\Pr(Y = 0) + \Pr(Y = 4) = 1/4 + 1/12 = 1/3 \approx .333$$

c) Use matrix algebra to determine the probability distribution for how many dollars Armando has after 4 coin flips. Be sure to show what matrices you multiply.

Armando's distribution after 4 flips is:  $(0 \ .500 \ .333 \ .167 \ 0)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^4 = (.4583$$

.0833 .0833 .0833 .2917),  
which can be written as:

$x$	0	1	2	3	4
$p(x)$	.4583	.0833	.0833	.0833	.2917

d) Use matrix algebra to determine the probability distribution for how many dollars Armando has after 32 coin flips.

Armando's distribution after 32 flips is:  $(0 \ .500 \ .333 \ .167 \ 0)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{32} = (.5833$$

.0000 .0000 .0000 .4167),  
which can be written as:

$x$	0	1	2	3	4
$p(x)$	.5833	.0000	.0000	.0000	.4167

e) What do these calculations reveal about Armando's probability of winning the \$4?

It appears that Armando's probability of winning the \$4 is very close to .4167.