STAT 325 Introduction to Probability Models Spring 2010

HW14 due Thur May 6

Topics: Absorption states, mean time to absorption, probabilities of entering absorption states

1. Reconsider (from HW13) the scenario that Armando has \$1 and Blanca has \$3. They play a game based on flipping a fair coin. If the coin lands heads, then Armando pays Blanca \$1. If the coin lands tails, then Blanca pays Armando \$1. They continue playing until one of them has all \$4. Let the random variable T represent the number of coin flips before the game ends.

a) What is the smallest possible value of T? Also determine its probability.

The smallest possible value of T is 1. $Pr(T = 1) = Pr(H_1) = .5$

b) Determine the probability that their game ends in 3 or fewer flips. (Be sure to indicate how you calculate this.)

$Pr(T \leq 3)$ is the sum of the first and last columns of the second row of the matrix P^3, where P =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It turns out that $P^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .625 & 0 & .25 & 0 & .125 \\ .25 & .25 & 0 & .25 & .25 \\ .125 & 0 & .25 & 0 & .625 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, so $Pr(T \leq 3) = .625 + .125 = .75$

c) Determine the probability that their game lasts for more than 6 flips.

$Pr(T > 6) = 1 - Pr(T \leq 6) = 1 -$ sum of 1st and last columns of 2nd row of P^6. Alternatively, $Pr(T > 6)$ = sum of 2nd, 3rd, and 4th columns of 2nd row of P^6. It turns out that P^6 =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .6875 & .0625 & 0 & .0625 & .1875 \\ .4375 & 0 & .1250 & 0 & .4375 \\ .1875 & .0625 & 0 & .0625 & .6875 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

, so $Pr(T > 6) = .0625 + .0625 = .125$

d) State the system of equations that must be solved to determine the expected value of the number of flips that the game lasts for.

We need to solve (I – Q)μ = **1**, where I is the 3×3 identity matrix, $Q = \begin{pmatrix} 0 & .5 & 0 \\ .5 & 0 & .5 \\ 0 & .5 & 0 \end{pmatrix}$, and **1** is a 3×1 vector of 1s.

e) Solve that system of equations to determine the expected value of the number of flips that the game lasts for.

The solution is: $\mu = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$. Starting with Armando having $1, the expected number of flips that the game lasts for is 3.

f) State the system of equations that must be solved to determine the probabilities that Armando/Blanca end up winning this game.

We need to solve (I – Q) F = R, where I and Q are as described above and $R = \begin{pmatrix} .5 & 0 \\ 0 & 0 \\ 0 & .5 \end{pmatrix}$.

g) Solve that system of equations to determine the probabilities that Armando/Blanca end up winning this game.

The solution is: $F = \begin{pmatrix} .75 & .25 \\ .50 & .50 \\ .25 & .75 \end{pmatrix}$. So, starting with Armando having $1, the probability is .75 that Blanca wins the game and .25 that Armadno wins the game.

h) Now suppose that Blanca is only willing to play for 3 coin flips. If the game is not over at that point, she will forfeit and Alberto will be declared the winner. Determine the probability that Blanca wins this game. (Be sure to indicate how you calculate this.)

Recall that $P^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .625 & 0 & .25 & 0 & .125 \\ .25 & .25 & 0 & .25 & .25 \\ .125 & 0 & .25 & 0 & .625 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, and we loo at the 2nd row because Armando starts with \$1. So, the probability is .625 that Blanca wins with 3 or fewer flips. If she is not willing to play longer, then Armando wins with probability 1 - .625 = .375.

2. Consider the following game: You start with \$*k*, where *k* is a positive integer no larger than 5, and you roll a fair six-sided die for each turn of the game. If the die lands on 1-5, you lose \$1. If the die lands on 6, then you are given enough dollars to bring your current total to \$5. (If you already have \$5 when you roll a 6, then you stay at \$5.) The game ends when you reach \$0. Consider your current total as a Markov chain.

a) Determine the transition probability matrix for this Markov chain.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 5/6 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 5/6 & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 5/6 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 5/6 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \end{pmatrix}$$

b) Identify the absorbing state(s).

There is only one absorbing state: \$0.

c) Identify the matrices I, Q, and R.

I is the 5×5 identity matrix. Q is the 5×5 matrix formed by deleting the first row and forst column of P, so $Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1/6 \\ 5/6 & 0 & 0 & 0 & 1/6 \\ 0 & 5/6 & 0 & 0 & 1/6 \\ 0 & 0 & 5/6 & 0 & 1/6 \\ 0 & 0 & 0 & 5/6 & 1/6 \end{pmatrix}$, $R = \begin{pmatrix} 5/6 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

d) Determine the expected number of rolls for which the game lasts, for all possible starting values *k*.

These expected values are the solution $\boldsymbol{\mu}$ to the system of equations $(I - Q)\boldsymbol{\mu} = \mathbf{1}$. The solution (to 3 decimal places) is:

$$\boldsymbol{\mu} = \begin{pmatrix} 2.488 \\ 4.562 \\ 6.290 \\ 7.730 \\ 8.930 \end{pmatrix}.$$