STAT 325 Introduction to Probability Models Spring 2010

HW16 due Tues May 11

Topics: Expected values, variance for continuous random variables

1. Let X denote the score on an exam for a randomly selected student in a course. Suppose that the pdf of X is given by: $f(x) = cx^2$, $0 < x < 100$, $f(x) = 0$ otherwise, where c is the appropriate constant.

a) Determine the value of c.

We need $\int_0^{100} cx^2 dx = 1$. The integral reduces to $1{,}000{,}000c/3$, so $c = 3/1{,}000{,}000 = .000003$.

b) Determine E(X).

$E(X) = \int_0^{100} x(cx^2)dx = c(100)^4/4 = 3(100)/4 = 75$

c) Determine Var(X) and SD(X).

$E(X^2) = \int_0^{100} x^2(cx^2)dx = c(100)^5/5 = 3(10{,}000)/5 = 6000$
$Var(X) = E(X^2) - [E(X)]^2 = 6000 - (75)^2 = 375$
$SD(X) = \text{sqrt}(375) \approx 19.365$

Now suppose that the course involves three exams, which are independent and all have the same pdf as above. But the third exam counts for double weight, so a student's overall score is a random variable $Y = X_1 + X_2 + 2X_3$.

d) Determine E(Y).

$E(Y) = E(X_1 + X_2 + 2X_3) = E(X_1) + E(X_2) + 2E(X_3) = 75 + 75 + 2(75) = 300$

e) Determine Var(Y) and SD(Y).

$Var(Y) = Var(X_1 + X_2 + 2X_3) = Var(X_1) + Var(X_2) + 2^2Var(X_3) = 375 + 375 + 4(375) = 2250$
$SD(Y) = \text{sqrt}(2250) \approx 47.434$

Suppose that a student misses the first exam because of illness, and so the professor says that the exam 2 score will be multiplied by 1.5 and the exam 3 score by 2.5. So, for this student, the overall score is $W = 1.5X_2 + 2.5X_3$.

f) Determine E(W) and SD(W). Comment on how they compare to E(Y) and SD(Y).

$E(W) = E(1.5X_2 + 2.5X_3) = 1.5EX_2) + 2.5E(X_3) = 1.5(75) + 2.5(75) = 300$, which is the same as $E(Y)$.

$Var(W) = Var(1.5X_2 + 2.5X_3) = (1.5)^2Var(X_2) + (2.5)^2Var(X_3) = 2.25(375) + 6.25(375) = 3187.5$, which is greater than $Var(Y)$.

$SD(W) = sqrt(3187.5) \approx 56.458$, which is greater than $SD(Y)$.

The student who misses the first exam has the same expected value for overall (adjusted) score as if he/she had not missed the exam, but the variability is greater.

2. Let the random variable Y have pdf $g(y) = c / y^3$ for $y > 1$ and $g(y) = 0$ for $y \le 1$, where c is the appropriate constant.

a) Determine the value of c.

We need $\int_1^\infty \frac{c}{y^3}dy = 1$, so we need $-\frac{c}{2y^2}\Big|_1^\infty = 1$, so we need $c/2 = 1$, so $c = 2$.

b) Determine E(Y).

$E(Y) = \int_1^\infty y \times \frac{c}{y^3}dy = \int_1^\infty \frac{2}{y^2}dy = -\frac{2}{y}\Big|_1^\infty = 2$

c) Determine the median of Y.

The median is the value (call it m) such that $\int_1^m \frac{2}{y^3}dy = .5$, so we need $-\frac{2}{2y^2}\Big|_1^m = .5$, so $1 - 1/m^2 = .5$, so $m = \sqrt{2} \approx 1.414$.

Now consider a more general random variable V with pdf $h(v) = c / v^r$ for $v > 1$ and $h(v) = 0$ for $v \le 1$, where c is the appropriate constant and r is some fixed value with $r > 2$.

d) Determine the value of the constant c, in terms of r.

We need $\int_1^\infty \frac{c}{v^r}dv = 1$, so we need $\frac{c}{(1-r)v^{r-1}}\Big|_1^\infty = 1$, so we need $c/(r-1) = 1$, so $c = r - 1$.

e) Verify that your answer to d) is consistent with your answer to a).

In part a), we used $r = 3$ and found $c = 2$, so $c = r - 1$ is consistent.

f) Determine E(V) as a function of r. Make sure that your answer is consistent with b).

$$E(V) = \int_1^{\infty} v \times \frac{r-1}{v^r} dv = \int_1^{\infty} \frac{r-1}{v^{r-1}} dv = \left. \frac{(r-1)}{(2-r)v^{r-2}} \right|_1^{\infty}$$ $= (r - 1) / (r - 2)$. This is consistent with b).