STAT 325 Introduction to Probability Models Spring 2010

HW9 due Wed Apr 21

Topics: Expected value, variance, properties of expected value and variance

1. Recall that a subgroup of 2 people is to be chosen at random from a committee consisting of 4 men (Abe, Ben, Chuck, Don) and 2 women (Ellen, Frannie). Let the random variable W = number of women selected.

a) Determine the expected value of W.

The pmf of W is: $\Pr(W = 0) = \dfrac{\binom{4}{2}\binom{2}{0}}{\binom{6}{2}} = 6/15$, $\Pr(W = 1) = \dfrac{\binom{4}{1}\binom{2}{1}}{\binom{6}{2}} = 8/15$, $\Pr(W = 2) = \dfrac{\binom{4}{0}\binom{2}{2}}{\binom{6}{2}} = 1/15$.

The expected value of W is: $E(W) = 0(6/15) + 1(8/15) + 2(1/15) = 10/15 = 2/3 \approx 0.667$.

b) Determine the probability that W equals its expected value.

Pr(W = 2/3) = 0 because it's impossible to select 2/3 of a woman.

c) Write a sentence interpreting what this expected value means.

If this random selection process were repeated a large number of times and the number of women recorded each time, the average number of women selected per repetition of the process would become very close to 2/3.

2. Suppose that you operate a small newsstand and that you must decide how many copies of a specialized magazine to order each week. Suppose that you pay $2 for each copy that you order and that you charge customers $5 per copy. Based on past experience, you believe that the number of customers requesting the magazine will be 2, 3, or 4, with the following probabilities:

Demand	2	3	4
Probability	.2	.5	.3

Let the random variable R represent your profit.

a) Suppose that you order 2 magazines. Determine the pmf of R and expected value of R. [*Hint*: You can't sell more magazines that you order in the first place.]

You are guaranteed to have demand for at least 2 magazines, so you will sell both (with probability 1). You pay \$4 to buy the two copies and then sell them for a total of \$10, so your profit is \$6 (with probability 1). In table form the pmf is:

r	6
p(*r*)	1

Then E(R) = 6(1) = 6.0

b) Repeat a), supposing that you order 3 magazines.

You pay \$6 to order the 3 copies. When the demand is only for 2 magazines (which has probability .2), you take in \$10 for a net profit of \$4. When the demand is for either 3 or 4 magazines (which has probability .8), you sell all 3 that you ordered and take in \$15 for a net profit of \$9. So, the pmf of R is:

r	4	9
p(*r*)	.2	.8

Then E(R) = 4(.2) + 9(.8) = 8.0

c) Repeat a), supposing that you order 4 magazines.

You pay \$8 to order the 4 copies. When the demand is only for 2 magazines (which has probability .2), you take in \$10 for a net profit of \$2. When the demand is for 3 magazines (which has probability .5), you take in \$15 for a net profit of \$7. When you sell all 4 magazines (which has probability .3), you take in \$20 for a net profit of \$12. So, the pmf of R is:

r	2	7	12
p(*r*)	.2	.5	.3

Then E(R) = 2(.2) + 7(.5) + 12(.3) = 7.5

d) How many magazines should you order, to maximize your expected profit?

Your expected profit is maximized by ordering 3 copies, which gives an expected profit of \$8.

3. As in Example 7-6, consider a best-of-three series between two sports teams, which means that the series ends as soon as one team has won two games. Let the random variable G represent the total number of games required to complete the series.

a) Determine the pmf of G and E(G) under the assumption that the two teams are equally matched (so the probability that team A defeats team B in any one game is .5) and the games are independent.

The sample space of possible outcomes is: $\{A_1A_2, B_1B_2, A_1B_2A_3, A_1B_2B_3, B_1A_2A_3, B_1A_2B_3\}$. The first 2 outcomes listed produce G = 2, and the last 4 outcomes listed produce G = 3. The probabilities are:

$\Pr(G = 2) = \Pr(A_1A_2 \cup B_1B_2) = \Pr(A_1A_2) + \Pr(B_1B_2) = \Pr(A_1)\Pr(A_2) + \Pr(B_1)\Pr(B_2)$ because of independence $= (.5)(.5) + (.5)(.5) = .5$

$\Pr(G = 3) = 1 - \Pr(G = 2) = 1 - .5 = .5$

Then $E(G) = 2(.5) + 3(.5) = 2.5$

b) Repeat a) but assuming that the probability is 2/3 that team A defeats team B in any one game, and that the games are independent.

$\Pr(G = 2) = \Pr(A_1A_2 \cup B_1B_2) = \Pr(A_1A_2) + \Pr(B_1B_2) = \Pr(A_1)\Pr(A_2) + \Pr(B_1)\Pr(B_2) = (2/3)(2/3) + (1/3)(1/3) = 5/9 \approx .556$

$\Pr(G = 3) = 1 - \Pr(G = 2) = 1 - 5/9 = 4/9 \approx .444$

Then $E(G) = 2(5/9) + 3(4/9) = 22/9 \approx 2.444$

c) Repeat a), but assuming that team A has a 2/3 probability of winning game #1, team B has a 2/3 probability of winning game #2, and both teams have a .5 probability of winning game #3. (For instance, this could mean that game #1 is played on team A's home field, game #2 is played on team B's home field, and game #3 is played on a neutral field.)

$\Pr(G = 2) = \Pr(A_1A_2 \cup B_1B_2) = \Pr(A_1A_2) + \Pr(B_1B_2) = \Pr(A_1)\Pr(A_2) + \Pr(B_1)\Pr(B_2) = (2/3)(1/3) + (1/3)(2/3) = 4/9 \approx .444$

$\Pr(G = 3) = 1 - \Pr(G = 2) = 1 - 4/9 = 5/9 \approx .556$

Then $E(G) = 2(4/9) + 3(5/9) = 23/9 \approx 2.556$

4. Consider a random variable (call it Y) that equals 1 with probability p and equals 0 with probability $1 - p$, where $0 < p < 1$.

a) Determine E(Y), as a function of p.

$E(Y) = 0(1-p) + 1(p) = p$

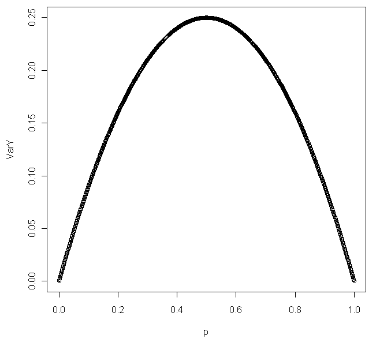
b) Determine $E(Y^2)$.

$E(Y^2) = 0^2(1-p) + 1^2(p) = p$

c) Determine Var(Y).

$Var(Y) = E(Y^2) – [E(Y)]^2 = p – p^2 = p(1- p)$

d) Graph Var(Y) as a function of *p*.



e) Use calculus to determine the value of *p* that maximizes Var(Y).

Let $f(p) = p – p^2$, so $f'(p) = 1 – 2p$ and setting $f'(p) = 0$ gives $1 - 2p = 0$ so solving gives $p = ½ = .5$. $f''(p) = -2$, so $p = .5$ does indeed produce a maximum for Var(Y).

f) What value(s) of *p* minimize(s) Var(Y)?

Var(Y) is minimized when $p = 0$ and $p = 1$, because Var(Y) = 0 there.

5. Let X be the result of rolling a single, fair, six-sided die. Then E(X) = 7/2 and Var(X) = 35/12.

Now suppose that you roll two fair, six-sided dice independently; let's call the results V and W. Suppose that I offer to play a game where your net profit is R = 2.5V – W.

a) Use properties of expected value and variance to determine E(R) and Var(R).

E(R) = E(2.5V – W) = 2.5E(V) – E(W) = 2.5(3.5) – 2.5 = 6.25

$Var(R) = Var(2.5V – W) = 2.5^2Var(V) + (-1)^2Var(W) = 6.25(35/12) + 1(35/12) \approx 21.146$

b) Determine Pr(R > 0).

There are only a few (*v*, *w*) outcomes for which $R \leq 0$: (1,3), (1,4), (1,5), (1,6), (2,5), (2,6). So, $Pr(R > 0) = 1 – Pr(R \leq 0) = 1 – 6/36 = 5/6 \approx 0.833$.

Now suppose that I offer you the choice of receiving X (the result of a single roll) or R.

c) Which option has the higher expected value?

E(R) = 6.25 and E(X) = 3.5, so R has the much higher expected value.

d) Which option has the higher probability that your profit will be positive?

X has a higher probability of a positive profit. $\Pr(X > 0) = 1$, whereas $\Pr(R > 0) = 5/6$.