STAT 325 Introduction to Probability Models Spring 2012

Investigation 3: Markov's Umbrellas (assigned on Mon Apr 30, due on Thur May 10)

You may work with one partner, submitting one report with both names, provided that you both contribute substantially to the work. Word-processed reports are strongly preferred to hand-written ones. Please integrate computer output into your report.

Suppose that Professor Markov owns k umbrellas, where k is an even positive integer. He starts out with $k/2$ umbrellas at home and $k/2$ umbrellas at his office. Every day he goes from home to office in the morning and returns to home from his office in the afternoon/evening. He only takes an umbrella with him when it's actually raining at the time. Let's assume that there's a p_{am} probability of rain when he leaves home every morning, independently from day to day, and p_{pm} probability of rain when he leaves his office every afternoon/evening, independently from day to day and also independent of the weather at home that morning. Let X_i be the number of umbrellas that Professor Markov has at home at the beginning of day i.

You will investigate three questions:

- What is the (steady-state) probability distribution of the number of umbrellas that he'll have at home?
- What is the (steady-state) expected value of the number of umbrellas that he'll have at home?
- What fraction of the time will he get wet (because it rains where he has 0 umbrellas)?

You will also investigate how the values of k, p_{am}, and p_{pm} affect the answers to these questions.

Two Umbrellas

First let k = 2, so Professor Markov has only 2 umbrellas, and he starts on the first day with 1 umbrella at home and 1 at his office. Also let $p_{am} = p_{pm} = .2$.

a) Determine the 1-step transition probability matrix P.

b) Determine P^2, and from that report the probability distribution of the number of umbrellas that he has at home after 2 days (i.e., at the start of the 3rd day).

c) Determine the expected value of the number of umbrellas that he has at home after 2 days.

d) Determine the probability that he gets wet on the 3rd day. [*Hint*: He gets wet when he has 0 umbrellas where he is *and* it rains at that time.]

e) Determine the probability distribution of the number of umbrellas that he has at home after 8 days. Also determine the expected value of this probability distribution.

f) Repeat e), considering the situation after 64 days.

g) Set up and solve the appropriate system of equations to determine the (exact) steady-state distribution of the number of umbrellas that he will have at home in the long run. Confirm that this distribution is close to your answer to f).

h) Use the steady-state probability distribution to determine the probability that Professor Markov gets wet (on a given day after the steady-state probabilities kick in).

More General Rain Probabilities

Continue to assume that Professor Markov has $k = 2$ umbrellas, but now keep the rain probabilities as p_{am} and p_{pm}.

i) Determine the 1-step transition probability matrix in terms of p_{am} and p_{pm}.

j) Let $p_{am} = .2$ and $p_{pm} = .4$. Determine the 1-step transition probability matrix P. Also approximate the steady-state probability distribution by calculating P^{64}. Also use the (approximate) steady-state probability distribution to calculate the expected number of umbrellas at home. Finally, calculate the probability that Professor Markov gets wet on the 65th day.

k) Repeat j), with $p_{am} = .2$ and $p_{pm} = .1$.

l) Comment on how your findings compare among the three special cases for (p_{am}, p_{pm}) that you've examined.

More Umbrellas

Assume again, as you did initially, that $p_{am} = p_{pm} = .2$. But now suppose that Professor Markov has $k = 8$ umbrellas.

m) Determine the 1-step transition probability matrix P.

n) Approximate the steady-state probability distribution by calculating P^m for a sufficiently large value of m. Report the value of m that you use as well as the (approximate) steady-state probability distribution. Also use that distribution to calculate the expected number of umbrellas at home. Finally, calculate the probability that Professor Markov gets wet on day (m+1).

o) Compare your answers to n) with your answer to h). Does Professor Markov considerably decrease his probability of getting wet by owning 8 umbrellas rather than 2?