1. Suppose that \( \Pr(A) = .6 \) and \( \Pr(B) = .8 \).

   a) Report the smallest and largest possible values of \( \Pr(A \cap B) \). Briefly justify one of these answers (you get to choose which), but do not bother to justify the other answer.

   The smallest possible value of \( \Pr(A \cap B) \) is .4. We know this because:
   \[
   \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = .6 + .8 - \Pr(A \cap B) \leq 1, \text{ so } \Pr(A \cap B) \geq .4.
   \]

   The largest possible value of \( \Pr(A \cap B) \) is .6. We know this because:
   \[
   (A \cap B) \subseteq A \text{ and } (A \cap B) \subseteq B, \text{ so } \Pr(A \cap B) \leq \Pr(A) = .6 \text{ and } \Pr(A \cap B) \leq \Pr(B) = .8.
   \]

   b) Report the smallest and largest possible values of \( \Pr(A \cup B) \). Briefly justify one of these answers (you get to choose which), but do not bother to justify the other answer.

   The smallest possible value of \( \Pr(A \cup B) \) is .8. We know this because:
   \[
   A \subseteq (A \cup B) \text{ and } B \subseteq (A \cup B), \text{ so } \Pr(A \cup B) \geq \Pr(A) = .6 \text{ and } \Pr(A \cup B) \geq \Pr(B) = .8.
   \]

   The largest possible value of \( \Pr(A \cup B) \) is 1. We know this because:
   \[
   \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = .6 + .8 - .4 = 1 \text{ when } \Pr(A \cap B) = .4, \text{ its smallest possible value.}
   \]

   c) Report the smallest and largest possible values of \( \Pr(A \mid B) \). Do not bother to justify your answers.

   The smallest possible value of \( \Pr(A \mid B) \) is .5. We know this because:
   \[
   \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \text{ so this equals its smallest possible value when } \Pr(A \cap B) \text{ equals its smallest value, which is when } \Pr(A \cap B) = .4, \text{ in which case } \Pr(A \mid B) = .4 / .8 = .5.
   \]

   The largest possible value of \( \Pr(A \mid B) \) is .75. We know this because:
   \[
   \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \text{ so this equals its largest possible value when } \Pr(A \cap B) \text{ equals its largest value, which is when } \Pr(A \cap B) = .6, \text{ in which case } \Pr(A \mid B) = .6 / .8 = .75.
   \]

2. a) An Italian engineering company named Ma-Vib employed 12 men and 18 women in early 2011. The company needed to select 15 employees to be laid off, and it turned out that the 15 selected were all women. If the selection of which 15 employees to lay off had been made at random, what is the probability that 15 women would have been selected? (Give an expression
for this probability, but do not bother to simplify your answer, and do not bother to explain your
answer.)

\[
\frac{\binom{18}{12}}{\binom{30}{15}} = \frac{\binom{18}{15}}{\binom{30}{15}} \approx .000005
\]

b) Generalize a) for a company with \( M \) male employees and \( F \) female employees that needs to
select \( k \) employees to be laid off (where \( k \) is a positive integer with \( k \leq F \)): Give an expression
for the probability that all \( k \) of the laid off employees turn out to be female, assuming that all
possible selections of \( k \) employees are equally likely. (Again do not bother to simplify your
answer or to explain.)

\[
\frac{\binom{F}{k} \binom{M}{0}}{\binom{F+M}{k}} = \frac{\binom{F}{k}}{\binom{F+M}{k}}
\]

c) Reconsider b) in the case where \( F < k < M + F \). Give an expression for the probability that all
\( F \) women are among the \( k \) employees that are laid off, again assuming that all possible selections
of \( k \) employees are equally likely. (Again do not bother to simplify your answer or to explain.)

\[
\frac{\binom{F}{k-F} \binom{M}{F}}{\binom{F+M}{k}} = \frac{\binom{M}{F-k}}{\binom{F+M}{k}}
\]

3. a) Suppose that four students (A, B, C, D) are each asked to name one of the other three with
whom they would like to be roommates. The rules are that if any pair of students names each
other, then that pair does indeed become roommates (a successful match). If all possible
responses of the four students are equally likely, independently from student to student, what is
the probability that two successful pairs of matched roommates are produced? Show your work
and/or justify your answer.

Each student has 3 choices, so there are a total of \( 3^4 = 81 \) possible choices. This is the
denominator of the probability.

The numerator is the number of ways that the choices can be made so two successfully matched
pairs. Student A has 3 choices, but then the other three only have one choice, because A’s
choice must choose A and the other two must choose each other.

This probability is therefore \( 3/81 = 1/27 \approx .037 \).

b) Now suppose that the same rules are used with a group of six students (A, B, C, D, E, F) who
are each asked to name one of the other five with whom they would like to be roommates. Again
assuming that all possible responses are equally likely, independently from student to student,
determine the probability that three successful pairs of roommates are produced. Again show
your work and/or justify your answer.
Each student now has 5 choices, so there are a total of $5^6 = 15,625$ possible choices. This is the denominator of the probability.

The numerator is the number of ways that the choices can be made so three successfully matched pairs. Student A has 5 choices, and then A’s choice must choose A. The next student then has 3 choices from the remaining 3, and then (as above) the other three only have one choice. There are therefore $5 \times 3 = 15$ choices that produce three successfully matched pairs.

This probability is therefore $15/15,625 = 3/3125 \approx 0.00096$.

4. Suppose that a computer program has a “bug” that is equally likely to be in any one of 3 sections of code. Call these sections A, B, and C, and assume that the program only has one bug. Now suppose that when you search a particular section to look for the bug, the probability is .8 that you will find the bug if it really is in that section (so the probability is .2 that you will miss the bug even when it is in the section that you search). Also assume that you never find a bug that is not really there.

Finally, suppose that you search section B and do not find the bug. Given this evidence, determine the updated probability that the bug is in section A, the updated probability that the bug is in section B, and the updated probability that the bug is in section C.

Use good notation and show your work.

Let $A$ be the event that the bug is really in section A, and similarly define events $B$ and $C$. Let $E$ be the event that a search of section B did not reveal the bug.

We know that $\Pr(A) = \Pr(B) = \Pr(C) = 1/3$.

We also know that $\Pr(E|A) = \Pr(E|C) = 1$ (because if the bug is really in A or C, then a search in B will definitely not find the bug) and $\Pr(E|B) = .2$ (because if the bug is really in B, then a search of B reveals it with probability .8).

Bayes’ Theorem gives that:

$$\Pr(A|E) = \frac{\Pr(E|A)\Pr(A)}{\Pr(E|A)\Pr(A) + \Pr(E|B)\Pr(B) + \Pr(E|C)\Pr(C)} = \frac{1\left(\frac{1}{3}\right)}{(1\left(\frac{1}{3}\right)+.2\left(\frac{1}{3}\right)+1\left(\frac{1}{3}\right))} = \frac{5}{11} \approx 0.4545$$

$$\Pr(A|E) = \frac{\Pr(E|B)\Pr(B)}{\Pr(E|A)\Pr(A) + \Pr(E|B)\Pr(B) + \Pr(E|C)\Pr(C)} = \frac{.2\left(\frac{1}{3}\right)}{(1\left(\frac{1}{3}\right)+.2\left(\frac{1}{3}\right)+1\left(\frac{1}{3}\right))} = \frac{1}{11} \approx 0.0909$$

$$\Pr(C|E) = \frac{\Pr(E|C)\Pr(C)}{\Pr(E|A)\Pr(A) + \Pr(E|B)\Pr(B) + \Pr(E|C)\Pr(C)} = \frac{1\left(\frac{1}{3}\right)}{(1\left(\frac{1}{3}\right)+.2\left(\frac{1}{3}\right)+1\left(\frac{1}{3}\right))} = \frac{5}{11} \approx 0.4545$$
5. Prove that if $A$ and $B$ are independent events, then $A^c$ and $B^c$ are also independent events. Justify all steps in your proof.

If $A$ and $B$ are independent, then $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ by one of the equivalent definitions.

To show that $A^c$ and $B^c$ are independent, we need to show that $\Pr(A^c \cap B^c) = \Pr(A^c) \times \Pr(B^c)$.

Now, $\Pr(A^c \cap B^c) = \Pr[(A \cup B)^c]$ by DeMorgan’s law

$= 1 – \Pr(A \cup B)$ by complement rule

$= 1 – [\Pr(A) + \Pr(B) – \Pr(A \cap B)]$ by addition rule

$= 1 – [\Pr(A) + \Pr(B) – \Pr(A) \times \Pr(B)]$ by multiplication rule for independent events, with our assumption that $A$ and $B$ are independent

$= 1 – \Pr(A) – \Pr(B) + \Pr(A) \times \Pr(B)$

$= 1 – \Pr(A) – \Pr(B)[1 – \Pr(A)]$

$= [1 – \Pr(A)][1 – \Pr(B)]$

$= \Pr(A^c) \times \Pr(B^c)$ by complement rule

So, we have shown that $\Pr(A^c \cap B^c) = \Pr(A^c) \times \Pr(B^c)$, so $A^c$ and $B^c$ are independent.

6. Consider again the situation in which two sports teams (A and B) play a series of games. Again assume that the outcomes of the games are independent and that team A has probability $p$ (with $0 \leq p \leq 1$) of defeating team B in any one game. Now suppose that the rules are that the series of games continues until one team has won two more games than the other team.

a) Determine the probability that team A wins the series, as a function of $p$. Use good notation, justify every step in your derivation, and simplify your answer as much as possible.

Let $W_i$ be the event that team A wins game $i$, $L_i$ be the event that team A loses game $i$, and let $A$ be the event that team A wins the series. Also let intersection be assumed when no symbol appears between two events.

Notice that there is one way for team A to win the series in 2 games: $W_1W_2$.

There are two ways for team A to win the series in 4 games: $W_1L_2W_3W_4$ or $L_1W_2W_3W_4$.

Similarly, there are four ways for team A to win the series in 6 games, because the teams have to split the first two games and split the second two games, and each split can happen in two ways.
In general, there are \(2^n\) ways for team A to win the series in \((2n + 2)\) games.

Therefore: \(\Pr(A) = p^2 \left[2[p(1-p)]p^2 + 4[p^2(1-p)^2]p^2 + 8[p^3(1-p)^3]p^2 + \ldots + 2^n[p^n(1-p)^n]p^2 + \ldots\right]
\]
\[= p^2 \sum_{n=0}^{\infty} 2^n[p(1-p)]^n = p^2 \sum_{n=0}^{\infty}[2p(1-p)]^n\]
\[= p^2 \frac{1}{1-2p(1-p)} \text{ because this is a geometric series}\]
\[= p^2 / [2p^2 - 2p + 1]\]

b) Produce a graph of this probability as a function of \(p\).

c) Compare the probability that team A wins this series to the probability that team A wins a conventional best-of-7 series. Comment on which type of series favors which team, depending (perhaps) on the value of \(p\). Show appropriate calculations and/or graphs to support your answer.

7. Consider again Poly’s urn scheme: Suppose that a box initially contains 3 red balls and 3 green balls. You select one ball at random from the box and then replace it along with \(k\) more of the same color, where \(k\) is a non-negative integer. Then you again select one ball at random from the box and again replace it along with \(k\) more of the same color. Then you again select one ball at random from the box.

Suppose that before this process begins, I will use a random mechanism to determine the value of \(k\) to be used. I will roll a pair of fair, four-sided dice and take the value of \(k\) to be the absolute value of the difference between the numbers rolled. But I will not show you the results of the dice rolls, so you will not know the value of \(k\).

a) Determine the (prior) probabilities for all possible values of \(k\). Use good notation and justify your answers. Prior to selecting any balls from the box, which value of \(k\) is most likely, and which is least likely?

\[
\Pr(k = 0) = \frac{4}{16} = \frac{1}{4} = .25
\]
\[
\Pr(k = 1) = \frac{6}{16} = \frac{3}{8} = .375
\]
\[
\Pr(k = 2) = \frac{4}{16} = \frac{1}{4} = .25
\]
\[
\Pr(k = 3) = \frac{2}{16} = \frac{1}{8} = .125
\]

b) Now suppose that all 3 balls selected from the box turn out to be the same color. Given this information (evidence), determine the updated probabilities for all possible values of \(k\). Use good notation and justify the steps in your calculation.

Let \(E\) denote the event that all 3 balls are the same color.
Then: \[ Pr(E \mid k = 0) = 2 \left( \frac{3}{6} \right) \left( \frac{3}{6} \right) \left( \frac{3}{6} \right) = \frac{1}{4} = .25 \]

\[ Pr(E \mid k = 1) = 2 \left( \frac{3}{6} \right) \left( \frac{4}{8} \right) \left( \frac{5}{8} \right) = \frac{5}{14} \approx .357 \]

\[ Pr(E \mid k = 2) = 2 \left( \frac{3}{6} \right) \left( \frac{5}{8} \right) \left( \frac{7}{10} \right) = \frac{7}{16} = .4375 \]

\[ Pr(E \mid k = 3) = 2 \left( \frac{3}{6} \right) \left( \frac{5}{9} \right) \left( \frac{9}{12} \right) = \frac{1}{2} = .5 \]

Then the denominator of Bayes’ Theorem (from the law of total probability) is:

\[ Pr(E) = Pr(E \mid k = 0)Pr(k = 0) + Pr(E \mid k = 1)Pr(k = 1) + Pr(E \mid k = 2)Pr(k = 2) + Pr(E \mid k = 3)Pr(k = 3) \]

\[ = \]

c) Given that all 3 balls selected from the box turn out to be the same color, which value of \( k \) is most likely, and which is least likely?

d) Now suppose that the 3 balls selected from the box turn out to be two of one color and one of the other color. Given this information (evidence), determine the updated probabilities for all possible values of \( k \). Use good notation and justify the steps in your calculation.

e) Given that the 3 balls selected from the box turn out to be two of one color and one of the other color, which value of \( k \) is most likely, and which is least likely?