1. In the game of roulette, a ball is spun on a wheel that has 38 numbered slots. These slots have colors as well as numbers: 18 are red, 18 black, and 2 green. You are allowed to bet either on a number or a color. If you bet $k$ dollars on a number, then you win $35k$ dollars if the ball lands on your number; otherwise you lose $k$ dollars. If you bet $k$ dollars on a color (red or black), then you win $k$ dollars if the ball lands in a slot of that color; otherwise, you lose $k$ dollars.

a) What is the expected (net) winnings from betting $k$ dollars on a number?

Let $X$ be the net winnings from betting $k$ dollars on a color. The probability distribution (pmf) of $X$ is:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-k$</th>
<th>$35k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>37/38</td>
<td>1/38</td>
</tr>
</tbody>
</table>

$E(X) = -k(37/38) + (35k)(1/38) = -k/19$

b) What is the expected (net) winnings from betting $k$ dollars on a color?

Let $Y$ be the net winnings from betting $k$ dollars on a number. The probability distribution (pmf) of $Y$ is:

<table>
<thead>
<tr>
<th>$y$</th>
<th>$-k$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(y)$</td>
<td>20/38</td>
<td>18/38</td>
</tr>
</tbody>
</table>

$E(Y) = -k(20/38) + (k)(18/38) = -k/19$

c) Determine the variance and standard deviation of the net winnings from betting $k$ dollars on a number.

$E(X^2) = (-k)^2 \times (37/38) + (35k)^2 \times (1/38) = 631k^2/19$

$Var(X) = E(X^2) - [E(X)]^2 = 631k^2/19 - (-k/19)^2 \approx 33.21k^2$

$SD(X) = \sqrt{33.21} \approx 5.76k$

d) Determine the variance and standard deviation of the net winnings from betting $k$ dollars on a color.

$E(Y^2) = (-k)^2 \times (20/38) + (k)^2 \times (18/38) = k^2$

$Var(Y) = E(Y^2) - [E(Y)]^2 = k^2 - (-k/19)^2 \approx 0.997k^2$


\[ \text{SD}(X) = \sqrt{0.997} \approx 0.9986k \]

e) Summarize what your calculations reveal about these two types of bets in roulette.

The two types of bets have the same expected value, but the variance is much greater with betting on a number compared to betting on a color. This means that in the long run you’ll lose the same amount with either bet, but you’ll have more variability in your winnings if you bet on a number.

2. a) Determine the variance and standard deviation of a random variable that has a uniform distribution on the interval \((a, b)\), where \(a\) and \(b\) are real numbers with \(a < b\).

The probability density function is: \( f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \).

\[
\begin{align*}
\text{E}(X) &= \int_a^b x \left( \frac{1}{b-a} \right) dx = \frac{b^2-a^2}{2} \times \frac{1}{b-a} = \frac{(a+b)}{2} \\
\text{E}(X^2) &= \int_a^b x^2 \left( \frac{1}{b-a} \right) dx = \frac{b^3-a^3}{3} \times \frac{1}{b-a} = \frac{(b-a)(b^2+ab+b^2)}{3} \times \frac{1}{b-a} = \frac{b^2+ab+b^2}{3} \\
\text{Var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2 = \frac{b^2+ab+b^2}{3} - \left( \frac{a+b}{2} \right)^2 = \ldots = \frac{(b-a)^2}{12}
\end{align*}
\]

b) Let \(c\) be a positive real number. Use your result from a) to determine the variance of a random variable that has a uniform distribution on the interval \((a + c, b + c)\).

The variance of a uniform distribution is the difference in endpoints squared and then divided by 12, which gives \((b - a)^2 / 12\).

In other words, shifting the interval by a constant amount does not change the variance.

3. Let \(X\) be a random variable with \(\text{E}(X) = \mu\) and \(\text{Var}(X) = \sigma^2\), and let \(c\) be an arbitrary constant. Show that \(\text{E}[(X - c)^2] = (\mu - c)^2 + \sigma^2\). Justify every step in your derivation.

\[
\begin{align*}
\text{E}[(X - c)^2] &= \text{E}(X^2 - 2cX + c^2) & \text{by multiplying out the square} \\
&= \text{E}(X^2) - 2c\text{E}(X) + c^2 & \text{using linear properties of expectation} \\
&= \text{E}(X^2) - \mu^2 + \mu^2 - 2c\mu + c^2 & \text{by adding and subtracting a constant} \\
&= \text{Var}(X) + (\mu - c)^2 & \text{by the short-cut formula for variance}
\end{align*}
\]

Not to turn in: Consider working on exercises 1-9 at the end of section 4.3 in the text. Notice that #5 is the third question on this assignment.