Stat 426 Assignment 5 (due Wed, Feb 20)
Topics: Maximum Likelihood Estimation, Method of Moments, Unbiased Estimators, Mean Squares Error

1. Suppose that a family of four wants to estimate the number of goldfish in its pond. Call this number *N*. The family decides to use a capture/recapture strategy for collecting data. They plan to capture a random sample of *A* fish, mark them so they can be identified later, and then release them back into the pond. After allowing sufficient time for these fish to circulate with the other fish, they will then capture a new, independent random sample of *n* goldfish and observe how many of those *n* goldfish are marked. Let the random variable X represent the number of goldfish captured in the second sample that are marked as having also been caught in the first sample.
a) Derive an expression for the method-of-moments estimator of *N*, in terms of *A*, *n*, and X.
Now suppose that the family captures 10 goldfish in the first sample and 12 goldfish in the second sample, of which 5 had been marked.
b) Determine the method-of-moments estimate of *N*.

2. Reconsider the previous problem. Again suppose that the family captures 10 goldfish in the first sample and 12 goldfish in the second sample, of which 5 had been marked.
a) Write out the likelihood function for *N*, and produce a sketch.
b) Determine the maximum likelihood estimate of *N*.

3. Let $X_1, \ldots, X_n$ constitute a random sample from a Gamma distribution with parameters α and β, where α is assumed to be known and β is to be estimated. Determine the maximum likelihood estimator of β.

4. Let $Y_1, \ldots, Y_n$ constitute a random sample from a population with probability density function $g(y \mid \theta) = (\theta + 1)(y^\theta)$ for $0 < y < 1$, where the parameter θ is greater than -1.
a) Determine the maximum likelihood estimator of θ.
b) Determine the method-of-moments estimator of θ.

5. Suppose that X_1, X_2, X_3 are i.i.d. according to a discrete uniform distribution between 1 and *N*, inclusive, where the parameter *N* is a positive integer.
a) Determine the method-of-moments estimator of *N*.
b) Calculate the method-of-moments estimate of *N*.
c) Determine the maximum likelihood estimator of *N*.

6. Let the random variable Y_i be the number of typographical errors on a page of a 400-page book (for $i = 1, \ldots, 400$), and suppose that the Y_i's are independent and identically distributed according to a Poisson distribution with parameter λ. Let the random variable X be the number of pages of this book that contain at least one typographical error. Suppose that you are told the value of X but are not told anything about the values of Y_i.
a) Determine the maximum likelihood estimator of λ through the following steps:
   i) Identify the probability distribution of X.
   ii) Write out the probability function of X in terms of the parameter λ.
   iii) Write out the likelihood function of lambda and also the log likelihood function of λ.

iv) Take the derivative of the log likelihood function with respect to λ.
v) Set that derivative equal to zero and solve for λ.

b) Suppose that the data reveal that 25 of the 400 pages contain a typographical error. What is the maximum likelihood estimate of λ?

c) Show that you could have used the invariance property of maximum likelihood estimators to determine the MLE of λ.

7. Let $X_1, \ldots, X_n$ constitute a random sample from a Beta distribution for which the parameters α and β are equal. Denote this common parameter by θ. Determine the method-of-moments estimator of θ. [*Hint*: The first moment may not be sufficient to produce an estimator.]

8. Let $Y_1, \ldots, Y_n$ constitute a random sample from a Normal distribution with mean known to equal 0 and unknown variance τ.

a) Determine the maximum likelihood estimator of τ.

b) Show that the method of moments produces the same estimator of τ as does maximum likelihood.

c) Determine whether or not this estimator is an unbiased estimator of τ.

d) Determine the maximum likelihood estimator of the standard deviation.

e) Determine whether or not the MLE estimator is an unbiased estimator of the standard deviation.

9. Let X_i denote the time that it takes student i to complete a take-home exam, and suppose that $X_1, \ldots, X_n$ constitute a random sample from an exponential distribution with parameter β. Let the mean $\mu = 1/\beta$ be the parameter of interest. For now consider estimators of μ having the form: $T_c = c \times \min\{X_i\}$.

a) Determine the value of c (perhaps as a function of n) for which T_c is an unbiased estimator of μ.

b) Determine the variance and MSE of this estimator from a), as a function of the parameter μ.

10. Reconsider the previous problem. Now consider estimating μ by waiting until everyone finishes the exam and calculating the average time.

a) Determine the bias, variance, and MSE of the sample mean $\overline{X}$ as an estimator of μ. (Express these in terms of the parameter μ rather than the parameter β.)

b) Calculate the ratio of the MSE of the sample mean $\overline{X}$ to the MSE of the unbiased estimator that you found in a) of the previous problem. Does one of these estimators have a smaller MSE than the other regardless of the value of β, and for any sample size n?